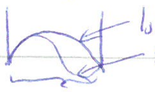
Session 3

ex. 5.1 given: $\lambda = 633\,nm$, $\Delta\nu_0^* = 1.7\cdot10^9\,Hz$, $L = 50\,cm$, $d = 3\,mm$

open resonator - 2 parallel mirrors



1st, 2nd modes $\Rightarrow L = \frac{n\lambda}{2} \rightarrow \lambda = \frac{2L}{n} \rightarrow \nu = \frac{c}{\lambda}$ $\Rightarrow$ $\boxed{\nu = \frac{cn}{2L}}$ $\rightarrow$ $\Delta\nu = \frac{c}{2L}$ ← eq. 5.1.3

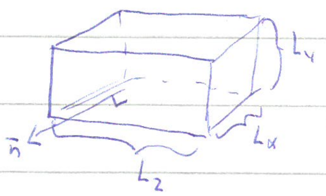
mode = time independent field amplitude distribution

$\Rightarrow$ # modes $\frac{\Delta\nu_0^*}{\Delta\nu} = \Delta\nu_0^* \cdot \frac{2L}{c} \approx 6$

closed resonators in chapter 2.2.1 considers perfect conducting closed resonator at thermal equilibrium

chapter 2.2 argues that shape is irrelevant at thermal equilibrium

$\Rightarrow$ can consider simple geometry - rectangular cavity



Solving $\begin{cases} \nabla^2 E - \frac{1}{c_n^2}\frac{\partial^2 E}{\partial t^2} = 0 \\ \bar{E}\times\bar{n} = \bar{0} \leftarrow \text{tangential field component vanishes on the wall!} \end{cases}$

↳ eq. 2.2.10, ex. $u_x = \cos(k_x x)\sin(k_y y)\sin(k_z z)$

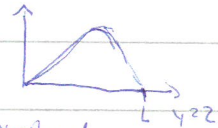
$k_i = \frac{n_i\pi}{L_i}$, $i = x,y,z$ & $n_i \geq 0$

the combination of cos and sin for u_x, u_y, u_z changes to satisfy $\bar{E}\times\bar{n} = \bar{0}$!

Simple ex. for visualization: $L_y = L_z = L$ & $n_x = 0$, $n_y = n_z = 1$ $\Rightarrow$ $u_x = \sin\left(\frac{\pi}{L}y\right)\sin\left(\frac{\pi}{L}z\right)$, where $y = z \rightarrow$

makes $u_y = u_z = 0$!

(s.t.) $u_x = \sin^2\left(\frac{\pi}{L}y\right)$ ← $y = z$



"similar" form as lowest order in the open resonator,

gives a feel for why there is $\frac{1}{L}$ instead of $\frac{1}{2L}$ as in the open resonator!

$\rho(\nu) = \frac{1}{V}\frac{dN}{d\nu} = \frac{8\pi\nu^2}{c_n}$ ← eq. 2.2.16

↑ $\frac{\#\text{modes}}{\text{volume}\cdot\text{frequency range}}$

$\Rightarrow N = V\Delta\nu_0^*\frac{8\pi\nu^2}{c_n^3} = \left[\nu = \frac{c}{\lambda},\ c_n = c \ (n=1),\ V = \pi\left(\frac{d}{2}\right)^2 L\right] \Rightarrow$

$\Leftrightarrow = \pi\left(\frac{d}{2}\right)^2 L \cdot \frac{8\pi\Delta\nu_0^*}{c\lambda^2} = \frac{d^2\pi^2}{\lambda^2}\cdot\underbrace{\frac{2L\Delta\nu_0^*}{c}}_{\#\text{ modes in open cavity}} \approx 1.2\cdot10^9 \gg 6$

$\Rightarrow$ laser resonators tend to be open to reduce the number of oscillating modes!

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open resonator $\Rightarrow$ diffraction losses - the modes are not entirely time independent but the relative field distribution can be time independent - eq. 5.1.4 $\bar{E}(\bar{r},t) = E_0\,\bar{u}(\bar{r})\,e^{-t/2\tau_c}$ ... (XX) ← losses

the mode has to reproduce itself after a roundtrip $\Leftrightarrow$ the field has to be in phase with itself after a roundtrip $\Rightarrow$

$$-2kL + \phi_{lm} = -2\pi n \quad (X) \leftarrow \text{"in phase condition"}$$

cavity length; $k = \frac{2\pi\nu}{c}$; phase because of propagation

phase from the mode $\rightarrow$ $E^{(1)}_{lm}(x,y) = K_{lm}\,e^{j\phi_{lm}}\,E^{(0)}_{lm}(x,y)$

damping, phase; $K_{lm}e^{j\phi_{lm}}$: eigenvalue; $E^{(1)}_{lm}$: field after 1 roundtrip; $E^{(0)}_{lm}$: start field

ex. e^{-j2kL} neg. phase factor

(X) & $k = \frac{2\pi\nu}{c}$ $\Rightarrow$ $\nu_{nlm} = \frac{c}{2L}\left(n + \frac{\phi_{lm}}{2\pi}\right)$ ← eq. 5.2.7

← depends on transverse mode!

$Q = 2\pi\frac{\text{stored energy}}{\text{energy lost in one cycle of oscillation}} = \left[(XX): E \propto e^{-\frac{t}{2\tau_c}} \rightarrow \phi \propto I \propto |E|^2 \propto e^{-\frac{t}{\tau_c}}\right] \Rightarrow$

photons (ϕ); life time (τ_c)

$\Leftrightarrow = 2\pi\cdot\frac{h\nu\phi}{h\nu\phi\cdot\frac{1}{\nu}\cdot\frac{1}{\tau_c}} = 2\pi\nu\tau_c = \left[\text{lorentzian} \Rightarrow \Delta\nu_c = \frac{1}{2\pi\tau_c}\right] = \frac{\nu}{\Delta\nu_c}$

$\frac{1}{\nu}$: time for 1 oscillation; $\frac{1}{\tau_c}$: portion of photons lost

low losses $\Rightarrow$ high Q!

unstable cavity = arbitrary ray bouncing between the mirrors will diverge indefinitely

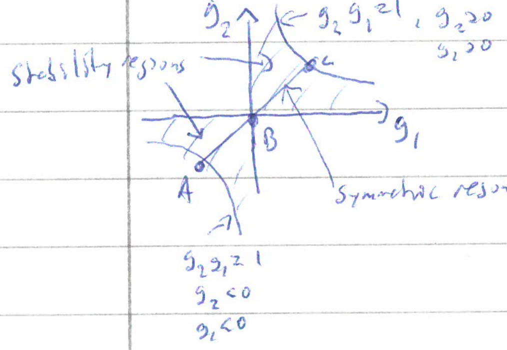
stable cavity = arbitrary ray bouncing between the mirrors will stay bounded between the mirrors

consider stable 2-mirror cavity $\begin{pmatrix} r_n \\ \theta_n \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^n \begin{pmatrix} r_0 \\ \theta_0 \end{pmatrix}$ ← matrix describing round trip; ← starting ray; ↑ ray after n round trips; $\left(\theta = \frac{dr}{dx}\right)$

stable if $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^n$ doesn't diverge when $n \to \infty$ ⇒ eq. 5.4.6 $-1 < \frac{A+D}{2} < 1$

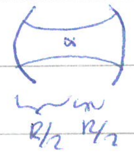
if the cavity doesn't have any other element than 2 mirrors; define $g_1 = 1 - \frac{L}{R_1}$, $g_2 = 1 - \frac{L}{R_2}$

⇒ eq. 5.4.6 ↦ eq. 5.4.11 $0 < g_1 g_2 < 1$

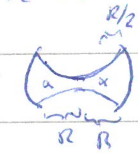


$f_{mirror} = \frac{R}{2}$

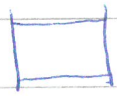
B: confocal $g = 1 - \frac{R}{R} = 0$



A: concentric $g = 1 - \frac{2R}{R} = -1$



C: plane $g = 1 - \frac{1}{\infty} = 1$



$\boxed{\alpha\alpha}$ consider ∞ aperture cavities & paraxial approx.

spot sizes for symmetric cavities:

on mirrors → $W = \sqrt{\frac{L\lambda}{\pi}}\left(\frac{1}{1-g^2}\right)^{1/4}$ ← eq. 5.5.10a

beam waist → $W_0 = \sqrt{\frac{L\lambda}{\pi}}\left(\frac{1+g}{4(1-g)}\right)^{1/4}$ ← eq. 5.5.10b

B: $W = \sqrt{\frac{L\lambda}{\pi}}$, $W_0 = \sqrt{\frac{L\lambda}{2\pi}}$ differ by $\frac{1}{\sqrt{2}}$

A: $W = \infty$, $W_0 = 0$ ← because of paraxial approx.*
↑ because of ∞ aperture (extent of mirrors)
*need to account for higher order diffraction terms when focusing light in concentric cavities!

eq. 5.2.7: $\nu_{nlm} = \frac{c}{2L}\left(n + \frac{\phi_{lm}}{2\pi}\right)$

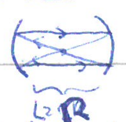
for 2-mirror cavities with $\boxed{\alpha\alpha}$-approximations we gets

$$\nu_{nlm} = \frac{c}{2L}\left[n + \frac{1+l+m}{\pi}\cos^{-1}\left(\pm\sqrt{g_1 g_2}\right)\right] \leftarrow \text{eq. 5.5.21}$$

C: $W = W_0 = \infty$ – infinite apertures – definition of plane wave

5.2 $L = 1\,m$, $\lambda = 514.5\,nm$, confocal resonator

a) spot size on mirrors & in the center? eq. 5.5.10 a,b → $W = \sqrt{\frac{L\lambda}{\pi}} \approx 400\,\mu m$, $W_0 = \sqrt{\frac{L\lambda}{2\pi}} \approx 283\,\mu m$



$n\,4R = n\lambda \Rightarrow \nu = \frac{c}{\lambda} = \frac{c}{\frac{4R}{n}} = \frac{nc}{4R}$ → $\Delta\nu = \frac{c}{4L} \approx 75\,MHz$

b, c) # non-degenerate modes within Doppler broadened width $\Delta\nu_0^* = 3.5\,GHz$?

eq. 5.5.21: $\nu_{nlm} = \frac{c}{2L}\left[n + \frac{1+l+m}{\pi}\underbrace{\cos^{-1}(0)}_{\frac{2\pi}{2}}\right] = \frac{c}{4L}(2n + 1 + l + m)$ ← $g_1 = g_2 = 0$ for confocal!

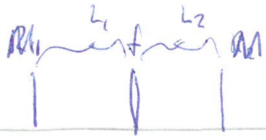
longitudinal modes – different n: keeping l, m the same → $\Delta\nu_{(n)lm} = \frac{c}{2L} = 150\,MHz$

transverse modes – different l, m: keeping n and l or m the same $\Delta\nu_{n(l)m} = \Delta\nu_{nl(m)} = \frac{c}{4L} = 75\,MHz$ ← smallest frequency difference!

$$\Rightarrow \left(\frac{\Delta\nu_{n(l)m}}{\Delta\nu_0^*}\right)^{-1} \approx 47$$

this is the number of unique frequencies within $\Delta\nu_0^*$, but they are all degenerate since different combinations of values for n, l, m gives the same values for ν_{nlm}!

5.8



To determine the spotsize & the stability need to know how the beam propagates through the cavity ⇒ start with roundtrip matrix!

$R=\infty$ mirror → $M_M = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ $\quad M_{L_1} = \begin{pmatrix}1&L_1\\0&1\end{pmatrix}$ $\quad M_{L_2} = \begin{pmatrix}1&L_2\\0&1\end{pmatrix}$ $\quad M_f^{\text{lens}} = \begin{pmatrix}1&0\\-1/f&1\end{pmatrix}$

table 4.1

starting from R_1: $M_{RT1} = M_{L_1} M_f M_{L_2} M_M M_{L_2} M_f M_{L_1} M_M = \dots = \frac{1}{f^2}\begin{pmatrix} f^2+2L_1L_2-2L_1f-2L_2f & 2(L_1f^2-L_1f-2L_1L_2f+L_1^2L_2+L_2f^2) \\ -2(f-L_2) & f^2+2L_1L_2-2L_1f-2L_2f \end{pmatrix}$

starting from f: $M_{RT2} = M_{L_1} M_M M_{L_1} M_f M_{L_2} M_M M_{L_2} M_f = \dots = \frac{1}{f^2}\begin{pmatrix} f^2-2L_2f-4L_1f+4L_1L_2 & 2L_1f^2-4L_1L_2f+2L_2f^2-2L_2f+f^2 \\ 2L_2-2f & \end{pmatrix}$

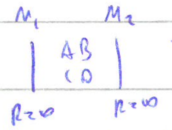
⌈MATLAB: syms L_1 L_2 f , define the matrices as above and write: M_{RT_1} = simplify($M_{L_1} M_f M_{L_2} M_M M_{L_2} M_f M_{L_1} M_M$)
e.g. M_M = [1,0;0,1]
M_{RT_2} = simplify($M_{L_1} M_M M_{L_1} M_f M_{L_2} M_M M_{L_2} M_f$)
insert * (multiplication) between all matrices
⌊ ⌋

The cavity is stable if $-1 < \frac{A+D}{2} < 1$ ← eq. 5.4.6 ⇒ $-1 < 1 - \frac{2(L_1+L_2)}{f} + \frac{2L_1L_2}{f^2} < 1$ (1)
for both M_{RT1} & M_{RT2} – starting point doesn't matter!

(1) $+1, \cdot f^2, \cdot\frac{1}{2}$ ⇒ $0 < f^2 - (L_1+L_2)f + L_1L_2 < f^2$ ⇔ $0 < (f-L_1)(f-L_2) < f^2$ ⇔ $0 < (1-\frac{L_1}{f})(1-\frac{L_2}{f}) < 1$

⇒ stable if L_1 and L_2 are less than f!

The spotsize on mirror 1 is obtained using: $\frac{1}{q_M} = \frac{1}{\infty} - j\frac{\lambda}{\pi w_M^2} = -j\frac{\lambda}{\pi w_M^2} \to q_M = \frac{\pi w_M^2}{-j\lambda}$ (2)
–//– mirror 2 is the same because of symmetry!



starting from M_2 gives the same roundtrip matrix as M_{RT1}!

eq. 4.7.4 $q = \frac{Aq_1+B}{Cq_1+D}$ (3) $q_1 = q$ after a roundtrip ⇒ $q = \frac{D-A}{2C} \pm \sqrt{\left(\frac{D-A}{2C}\right)^2 + \frac{B}{C}}$

for the mirror we use M_{RT1} where $A=D$ ⇒ $q_M = \pm\sqrt{\frac{B}{C}} \to w_M = \sqrt{\frac{-j\lambda}{\pi}\left(\pm\sqrt{\frac{B}{C}}\right)}$ ⇒

$$w = \sqrt{\frac{-j\lambda}{\pi}\cdot\left(\pm\sqrt{\frac{2(L_1f^2-L_1f-2L_1L_2f+L_1^2L_2+L_2f^2)}{-2(f-L_2)}}\right)}$$

choose signs to get $w_M > 0$ & w_M real!

The spotsize on the lens can be obtained as above using M_{RT2} instead or using (2) & (3) after propagating q_M through L_1: (3): $\frac{1}{q_f} = \frac{C+D/q_M}{A+B/q_M} = \left[\begin{pmatrix}A&B\\C&D\end{pmatrix} = \begin{pmatrix}1&L_1\\0&1\end{pmatrix}\right] = \frac{1/q_M}{1+L_1/q_M}$ →
← easier!
M_{L_1}

⇒ $\frac{-j\lambda}{\pi w_f^2} = \frac{-j\frac{\lambda}{\pi w_M^2}}{1+\frac{jL_1\lambda}{\pi w_M^2}} \cdot \frac{1-\frac{jL_1\lambda}{\pi w_M^2}}{1-j\frac{L_1\lambda}{\pi w_M^2}} = \dots$ ⇒ $w_f = w_M\sqrt{1+\frac{L_1^2\pi^2 w_M^2}{\lambda^2}}$

↖ not purely imaginary ⇒ $R \neq \infty$

Real resonators have finite apertures – such as mirror dimensions ⇒ changes field distribution from Hermite-Gaussian. Fox & Li studied losses for modes (TEM_{nm}) in cavities with different g-numbers & Fresnel numbers, $N = \frac{a^2}{L\lambda}$ ← aperture radius
eq. 5.5.26 ↗ L length of resonator

Fig 5.13

for symmetric resonator, eq. 5.5.10a for mirror spotsizes
$w = \sqrt{\frac{L\lambda}{\pi}}\left(\frac{1}{1-g^2}\right)^{1/4}$ $\quad \lambda = 514$ nm, $L = 1$ m → $g=0 \Rightarrow w \approx 0.4$ mm
$g = 0.9 \to w \approx 0.61$ mm
doesn't change much

$N = \frac{a^2}{L\lambda} = \left[L\lambda = w_c^2\pi\right] = \frac{a^2}{w_c^2\pi}$

$w_c = [g=0] = \sqrt{\frac{L\lambda}{\pi}} \Rightarrow \boxed{L\lambda = w_c^2\pi}$
confocal

fraction of aperture radius to spotsize with infinite aperture

lower diffraction loss for higher values of N (bigger mirrors relative to spotsize)
– TEM_{00} has lower losses than higher order modes

5.10 $\lambda = 630\,nm$, $g = 2\cdot10^{-2}$ per pass, $R_1 = R_2 = 10\,m$, $L = 1m$, mirror aperture to suppress TEM_{01}? and lase on TEM_{00}

the TEM_{01} losses has to exceed the gain $\Rightarrow$ be greater than 2%

in Fig 5.13 b) read off curve for $g = 1 - \frac{L}{R} = 0.9 \Rightarrow N \approx 3$ for 2% losses TEM_{01}

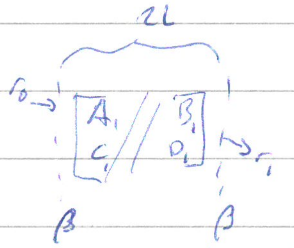
compare losses for TEM_{00} in Fig. 5.13 a) at $N \approx 3 \rightarrow$ less than 1%

lower losses for TEM_{01} will force the laser to operate with TEM_{00}

$\rightarrow N = \frac{a^2}{\lambda L} \rightarrow a = \sqrt{N\lambda L} \approx 1.4\,mm$

eq. 5.5.26

5.13 Fig 5.8d



Show: $0 < A_1 D_1 < 1$, $-1 < B_1 C_1 < 0$, $\frac{B_1 D_1}{A_1 C_1} < 0$ (for stable resonator) $\Rightarrow q$ purely imaginary

round trip: $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}_{rev} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ (plane mirror)

using (x) & $A_1 D_1 - B_1 C_1 = 1$ $\left(\frac{n_1}{n_2}\right)$ ← general

p. 173 true when input & output ray are in the same material (refractive index)

$\begin{pmatrix} r_1 \\ \frac{dr_1}{dx} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} r_0 \\ \frac{dr_0}{dx} \end{pmatrix} \rightarrow \begin{pmatrix} r_0 \\ \frac{dr_0}{dx} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} r_1 \\ \frac{dr_1}{dx} \end{pmatrix} \rightarrow$

$\Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} r_0 \\ \frac{dr_0}{d(-x)} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} r_1 \\ \frac{dr_1}{d(-x)} \end{pmatrix}$ ← backward $-x$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow$

$\Leftrightarrow (x) \begin{pmatrix} r_0 \\ \frac{dr_0}{d(-x)} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}_{rev}} \begin{pmatrix} r_1 \\ \frac{dr_1}{d(-x)} \end{pmatrix}$

eq. 5.5.5

$\Rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 2A_1D_1 - 1 & 2B_1D_1 \\ 2A_1C_1 & 2A_1D_1 - 1 \end{pmatrix}$

stable: $-1 < \frac{A+D}{2} < 1$ ← eq. 5.4.6

$\rightarrow -1 < \frac{2(2A_1D_1 - 1)}{2} < 1 \rightarrow 0 < 2A_1D_1 < 2 \rightarrow \underline{0 < A_1 D_1 < 1}$

$A_1D_1 - B_1C_1 = 1 \Rightarrow A_1D_1 = 1 + B_1C_1 \Rightarrow 0 < 1 + B_1C_1 < 1 \rightarrow -1 < B_1C_1 < 0$

eq. 5.5.2: $q = \frac{Aq + B}{Cq + D} \rightarrow q = \frac{D-A}{2C} \pm \sqrt{\left(\frac{D-A}{2C}\right)^2 + \frac{B}{C}} = [D = A] = \pm\sqrt{\frac{B}{C}} \Rightarrow$

$\Leftrightarrow = \pm\sqrt{\frac{2B_1D_1}{2A_1C_1}} = \pm\sqrt{\frac{B_1D_1}{A_1C_1} \cdot \frac{C_1}{C_1} \cdot \frac{D_1}{D_1}} = \pm\sqrt{\frac{B_1C_1 \cdot D_1^2}{A_1D_1 \cdot C_1^2}} \Leftrightarrow \pm\sqrt{\frac{(<0)\cdot(\geq 0)}{(\geq 0)\cdot(\geq 0)}} = \pm\sqrt{(<0)}$

purely imaginary!

11.8 given: $\lambda = 1.064\,nm$, $D_{FWHM} = 4\,mm$, $\theta = 3\,mrad$, M^2?

gauss intensity: $I = I_0 e^{\frac{-2r^2}{w_0^2}}$, FWHM: $\frac{I}{I_0} = \frac{1}{2} = e^{\frac{-2r_{hm}^2}{w_0^2}} \rightarrow \ln 2 = \frac{2r_{hm}^2}{w_0^2} \rightarrow r_{hm} = \frac{D_{FWHM}}{2} = w_0\sqrt{\frac{\ln 2}{2}} \rightarrow$

$\Leftrightarrow w_0 = \frac{D_{FWHM}}{\sqrt{2\ln 2}}$ eq. 4.7.13a

for gauss: $w = w_0\sqrt{1 + \left(\frac{z\lambda}{\pi w_0^2}\right)^2}$, far field ($z$ very large) $\Rightarrow \theta \approx \frac{w}{z} = \frac{w_0 \cdot \frac{z\lambda}{\pi w_0^2}}{z} \Rightarrow$

$\Leftrightarrow = \frac{\lambda}{\pi w_0}$, for modded gaussian: $\lambda \rightarrow M^2\lambda \Rightarrow \theta = \frac{M^2\lambda}{\pi w_0} \Rightarrow M^2 = \frac{\theta\pi w_0}{\lambda} = \frac{\theta\pi}{\lambda}\frac{D_{FWHM}}{\sqrt{2\ln 2}} \approx 30$