KTH ROYAL INSTITUTE
OF TECHNOLOGY

Wave Optics and Gaussian Beams

Ruslan Ivanov
OFO/ICT

Outline

- Differential approach: Paraxial Wave equation
- Integral approach: Huygens’ integral
- Gaussian Spherical Waves
- Higher-Order Gaussian Modes
  - Lowest Order Mode using differential approach
  - The ”standard” Hermite Polynomial solutions
  - The ”elegant” Hermite Polynomial solutions
  - Astigmatic Mode functions
- Gaussian Beam Propagation in Ducts
- Numerical beam propagation methods

The paraxial wave equation

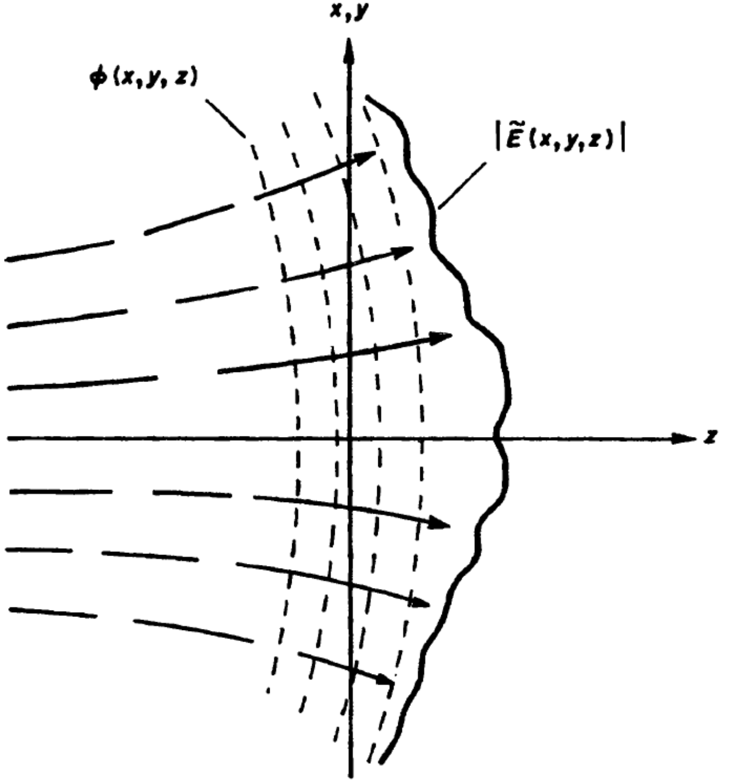
EM field in free space

$$\left[\nabla^2 + k^2\right] \tilde{E}(x,y,z) = 0$$

Extracting the primary propagation factor:

$$\tilde{E}(x,y,z) \equiv \tilde{u}(x,y,z)e^{-jkz}$$

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} + \frac{\partial^2 \tilde{u}}{\partial z^2} - 2jk\frac{\partial \tilde{u}}{\partial z} = 0$$



Transverse amplitude and phase variation of a paraxial optical wave.

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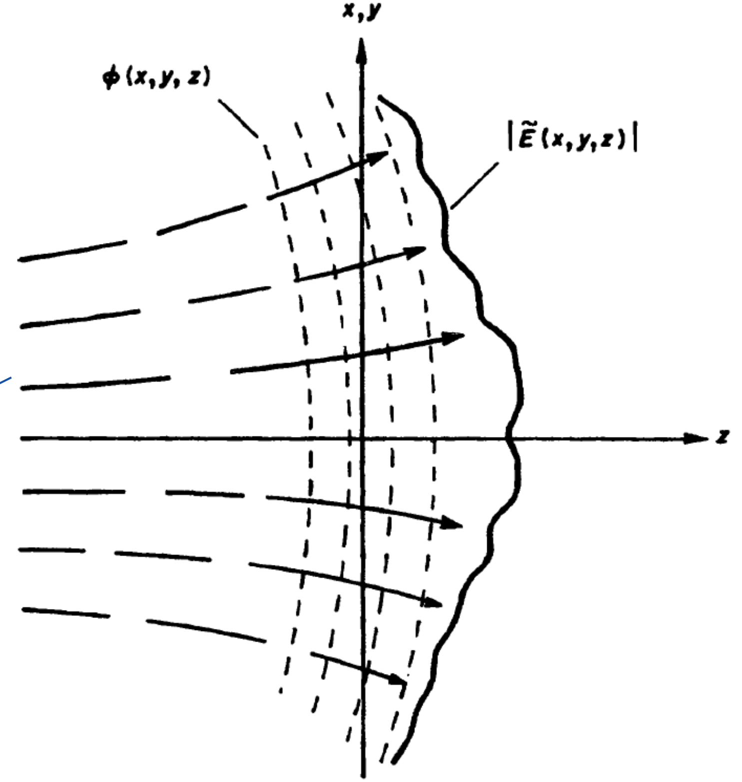
Paraxial approximation:

$$\left|\frac{\partial^2 \tilde{u}}{\partial z^2}\right| \ll \left|2k\frac{\partial \tilde{u}}{\partial z}\right|, \left|\frac{\partial^2 \tilde{u}}{\partial x^2}\right|, \left|\frac{\partial^2 \tilde{u}}{\partial y^2}\right|$$



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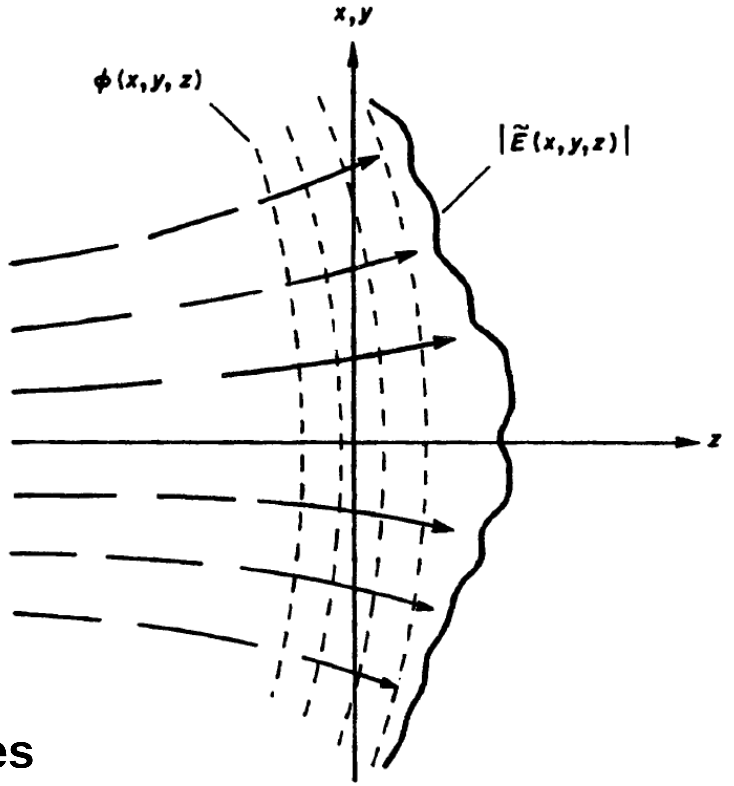
The paraxial wave equation then becomes

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} - 2jk\frac{\partial \tilde{u}}{\partial z} = 0$$



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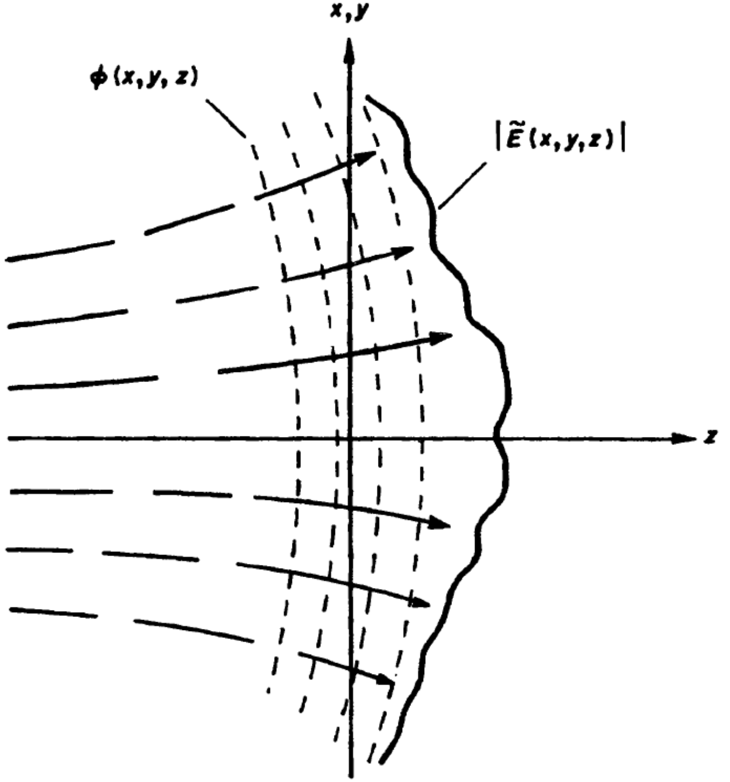
The paraxial wave equation

$$\nabla_t^2 \tilde{u}(s, z) - 2jk\frac{\partial \tilde{u}(s, z)}{\partial z} = 0$$

, where $s \equiv (x, y)$ - transverse coordinates

∇_t^2 - Laplacian operator in theses coordinates



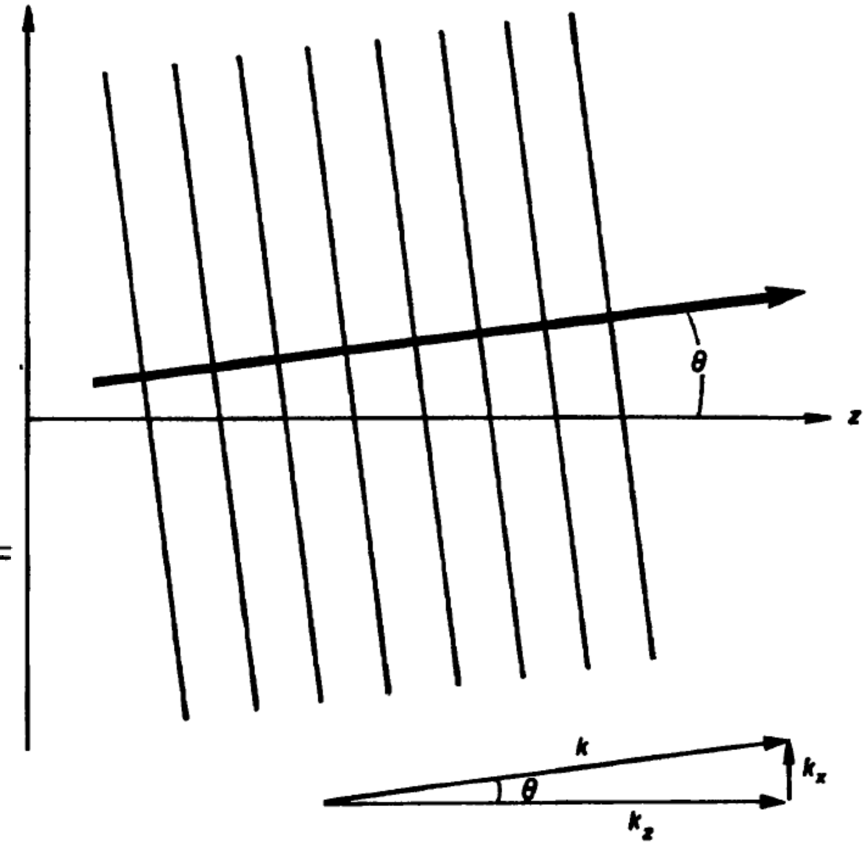
Transverse amplitude and phase variation of a paraxial optical wave.

Validity of the Paraxial Approximation

$$\frac{\partial \tilde{u}(\boldsymbol{s}, z)}{\partial z} = -\frac{j}{2k}\nabla_t^2 \tilde{u}(\boldsymbol{s}, z)$$

Arbitrary optical beam can be viewed as a superposition of plane wave components travelling at various angles to z axis

$$\tilde{E}(x,z) = \exp[-jkx\sin\theta - jkz\cos\theta] = $$
$$= \tilde{u}(x,z)e^{-jkz}$$



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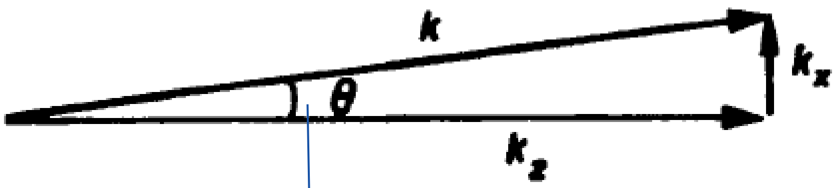

$$\tilde{E}(x,z) = \exp[-jkx\sin\theta - jkz\cos\theta] =$$
$$= \tilde{u}(x,z)e^{-jkz}$$

$\theta \ll 1$

The reduced wave amplitude

$$\tilde{u}(x,z) = \exp[-jkx\sin\theta + jkz(1-\cos\theta)] \approx \exp\left[-jk\theta x + jk\frac{\theta^2 z}{2}\right]$$

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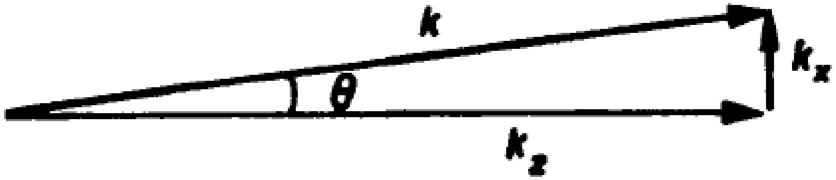

The reduced wave amplitude

$$\tilde{u}(x,z) \approx \exp\left[-jk\theta x + jk\frac{\theta^2 z}{2}\right]$$

$$-j\frac{2k}{\tilde{u}}\frac{\partial \tilde{u}}{\partial z} = +2k^2(1-\cos\theta) \approx k^2\theta^2$$

$$\frac{1}{\tilde{u}}\frac{\partial^2 \tilde{u}}{\partial x^2} = -k^2\sin^2\theta \approx -k^2\theta^2$$

$$\frac{1}{\tilde{u}}\frac{\partial^2 \tilde{u}}{\partial z^2} = -k^2(1-\cos\theta)^2 \approx -\frac{k^2\theta^4}{4}$$

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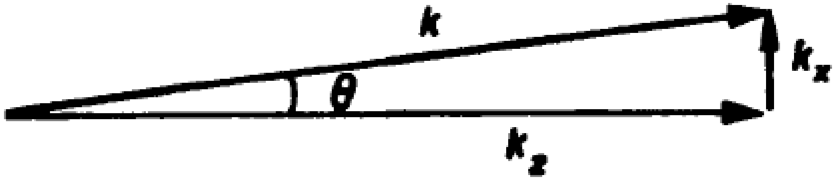

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To remind: Paraxial approximation

$$\left|\frac{\partial^2 \tilde{u}}{\partial z^2}\right| \ll \left|2k\frac{\partial \tilde{u}}{\partial z}\right|, \left|\frac{\partial^2 \tilde{u}}{\partial x^2}\right|, \left|\frac{\partial^2 \tilde{u}}{\partial y^2}\right|$$

$\theta^2/4 \ll 1$, i.e. $\theta < 0.5$ rad

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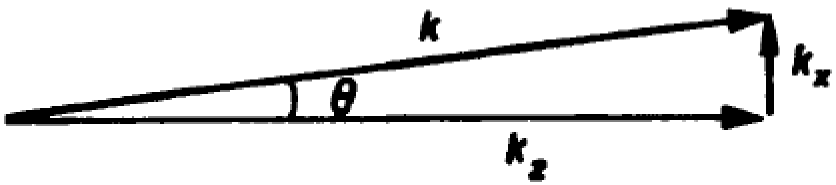

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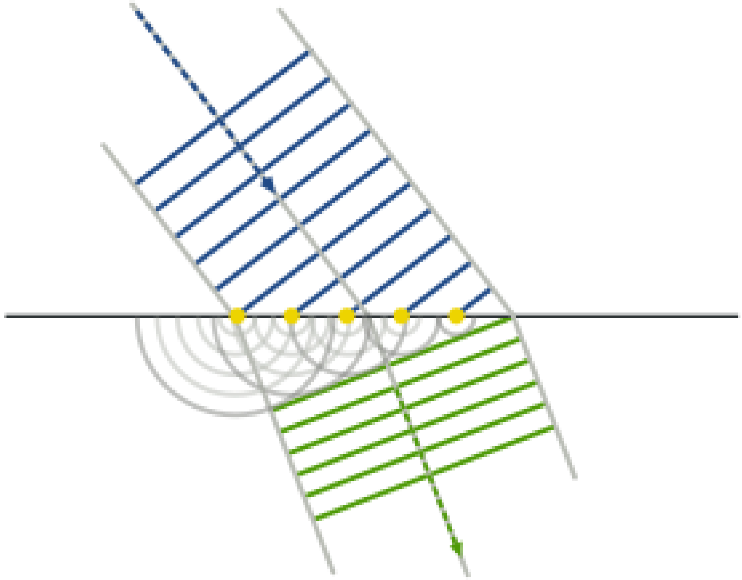
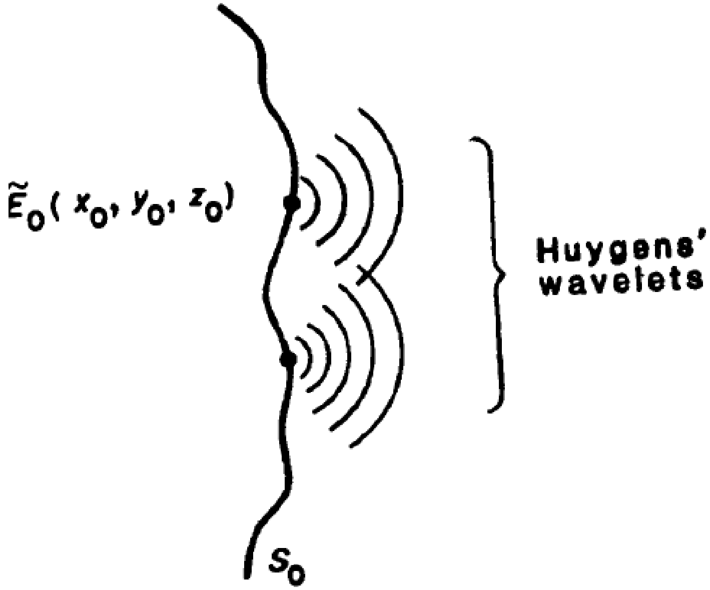
Paraxial optical beams can diverge at cone angles up to ≈30 deg before significant corrections to approximation become necessary

Huygens' Integral: Huygens' principle

"Every point which a luminous disturbance reaches becomes a source of a spherical wave; the sum of these secondary waves determines the form of the wave at any subsequent time"



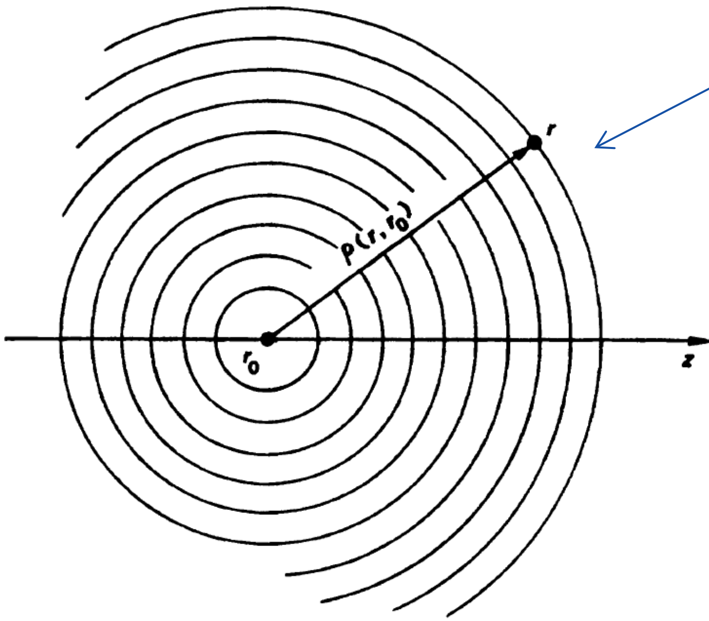


$$\tilde{E}(\boldsymbol{r};\boldsymbol{r}_0) = \frac{\exp[-jk\rho(\boldsymbol{r},\boldsymbol{r}_0)]}{\rho(\boldsymbol{r},\boldsymbol{r}_0)} \rightarrow \text{WE}$$

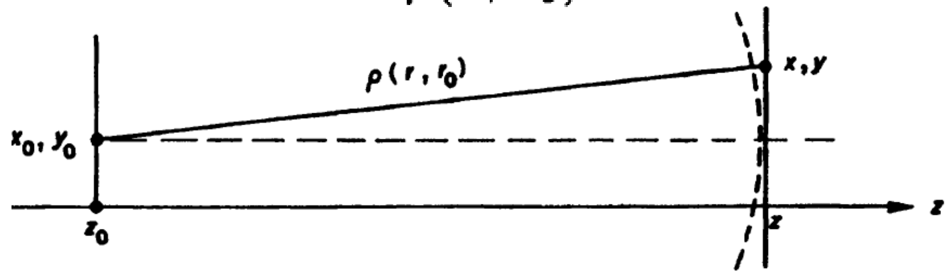
, where

$$\rho(\boldsymbol{r},\boldsymbol{r}_0) \equiv \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$$

Huygens' Integral: Fresnel approximation



$$\tilde{E}(\boldsymbol{r};\boldsymbol{r}_0) = \frac{\exp[-jk\rho(\boldsymbol{r},\boldsymbol{r}_0)]}{\rho(\boldsymbol{r},\boldsymbol{r}_0)}$$



Fresnel approximation:

$$\rho(\boldsymbol{r},\boldsymbol{r}_0) = z - z_0 + \frac{(x-x_0)^2 + (y-y_0)^2}{2(z-z_0)} + \cdots$$

Paraxial-spherical wave

$$\tilde{E}(x,y,z) \approx \frac{1}{z-z_0} \exp\left[-jk(z-z_0) - jk\frac{(x-x_0)^2 + (y-y_0)^2}{2(z-z_0)}\right]$$

$$\tilde{u}(x,y,z) = \frac{1}{z-z_0} \exp\left[-jk\frac{(x-x_0)^2 + (y-y_0)^2}{2(z-z_0)}\right]$$

→ PWE

Huygens' Integral

Huygens' principle

$$\tilde{E}(\boldsymbol{s}, z) = \frac{j}{\lambda} \int\int_{S_0} \tilde{E}_0(\boldsymbol{s}_0, z_0) \frac{\exp[-jk\rho(\boldsymbol{r}, \boldsymbol{r}_0)]}{\rho(\boldsymbol{r}, \boldsymbol{r}_0)} \cos\theta(\boldsymbol{r}, \boldsymbol{r}_0)\, d\boldsymbol{S}_0$$

,where $\rho(\mathbf{r},\mathbf{r_0})$ – distance between source and observation points
dS_0 – incremental element of surface are at $(\mathbf{s_0},\mathbf{z_0})$
$\cos\theta\ (\mathbf{r},\mathbf{r_0})$ – obliquity factor
j/λ – normalization factor

Huygens' Integral

Huygens' integral

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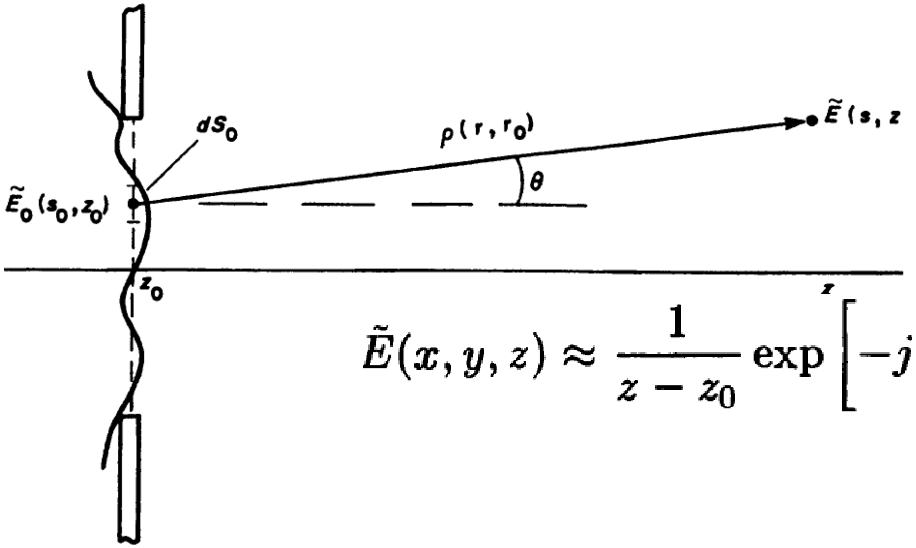


θ<<1 ➡ cosθ≈1

Spherical wave

⬇

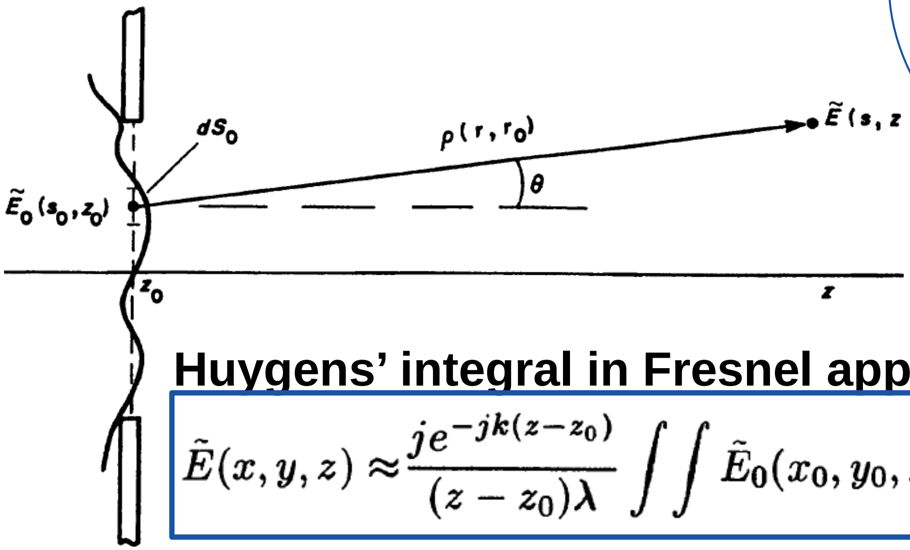
Paraxial-spherical wave

$$\tilde{E}(x, y, z) \approx \frac{1}{z - z_0} \exp\left[-jk(z - z_0) - jk\frac{(x - x_0)^2 + (y - y_0)^2}{2(z - z_0)}\right]$$

Huygens' Integral

Huygens' integral

$$\tilde{E}(\boldsymbol{s},z) = \frac{j}{\lambda}\int\int_{S_0} \tilde{E}_0(\boldsymbol{s}_0,z_0)\,\frac{\exp[-jk\rho(\boldsymbol{r},\boldsymbol{r}_0)]}{\rho(\boldsymbol{r},\boldsymbol{r}_0)}\,\cos\theta(\boldsymbol{r},\boldsymbol{r}_0)\,d\boldsymbol{S}_0$$



θ<<1 ➡ cosθ≈1

Spherical wave ➡ Paraxial-spherical wave

Huygens' integral in Fresnel approximation

$$\tilde{E}(x,y,z) \approx \frac{je^{-jk(z-z_0)}}{(z-z_0)\lambda}\int\int \tilde{E}_0(x_0,y_0,z_0)\exp\left[-jk\frac{(x-x_0)^2+(y-y_0)^2}{2(z-z_0)}\right]dx_0dy_0$$

, or the reduced wavefunction (with $L=z-z_0$)

$$\tilde{u}(x,y,z) = \frac{j}{L\lambda}\int\int \tilde{u}_0(x_0,y_0,z_0)\exp\left[-jk\frac{(x-x_0)^2+(y-y_0)^2}{2L}\right]dx_0dy_0$$

Huygens' Integral

Huygens' integral in Fresnel approximation

$$\tilde{u}(x,y,z) = \frac{j}{L\lambda} \int\int \tilde{u}_0(x_0,y_0,z_0) \exp\left[-jk\frac{(x-x_0)^2+(y-y_0)^2}{2L}\right] dx_0 dy_0$$

General form:

$$\tilde{u}(\boldsymbol{s},z) = \int\int \tilde{K}(\boldsymbol{r},\boldsymbol{r}_0)\,\tilde{u}_0(\boldsymbol{s}_0,z_0)\,d\boldsymbol{s}_0$$

$$\tilde{K}(\boldsymbol{r},\boldsymbol{r}_0) = \tilde{K}_1(x-x_0) \times \tilde{K}_1(y-y_0)$$ - Huygens kernel

$$\tilde{K}_1(x-x_0) = \sqrt{\frac{j}{L\lambda}} \exp\left[-j\frac{\pi(x-x_0)^2}{L\lambda}\right]$$ - 1D kernel

cilindrical wave

an initial phase shift of the Huygens' wavelet compared to the actual field value at the input point

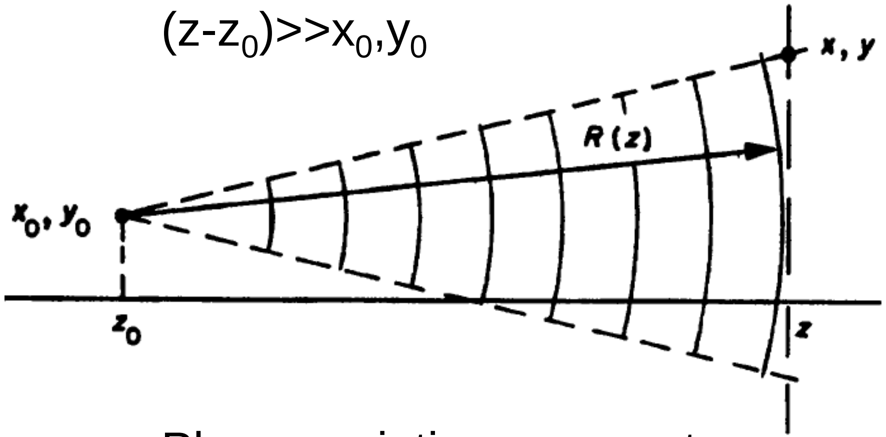
Then, if u_0 can be separated

$$\tilde{u}(x,z) = \sqrt{\frac{j}{L\lambda}} \int \tilde{u}_0(x_0,z_0) \exp\left[-j\frac{\pi(x-x_0)^2}{L\lambda}\right] dx_0$$

- 1D Huygens-Fresnel integral

Gaussian spherical waves

$(z-z_0)>>x_0,y_0$



Paraxial approximation

$$\tilde{u}(x,y,z) = \frac{1}{z-z_0}\exp\left[-jk\frac{(x-x_0)^2+(y-y_0)^2}{2(z-z_0)}\right]$$

$$= \frac{1}{R(z)}\exp\left[-jk\frac{(x-x_0)^2+(y-y_0)^2}{2R(z)}\right]$$

Phase variations across transversal plane

$$\phi(x,y,z) \equiv k\frac{(x-x_0)^2+(y-y_0)^2}{2(z-z_0)} = \frac{\pi}{\lambda}\frac{(x-x_0)^2+(y-y_0)^2}{R(z)}$$

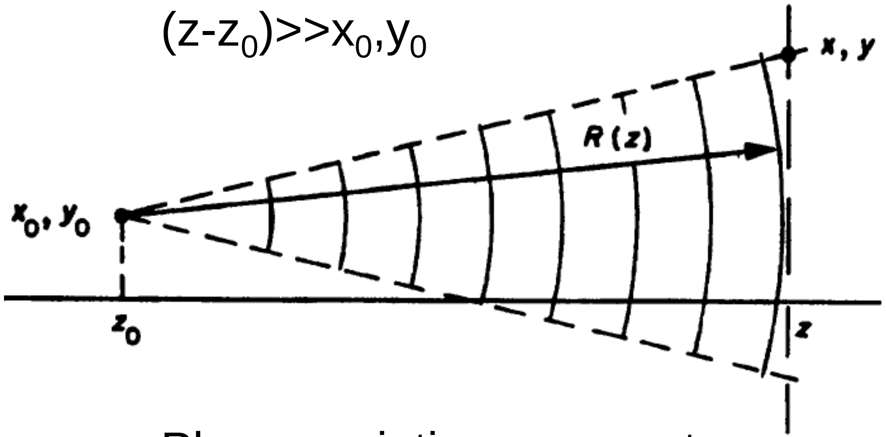
The radius of curvature of the wave plane

$$R(z) = R_0 + z - z_0$$

Quadratic phase variation represents paraxial approximations, so it is valid close to z axis

Gaussian spherical waves

$(z-z_0) >> x_0, y_0$



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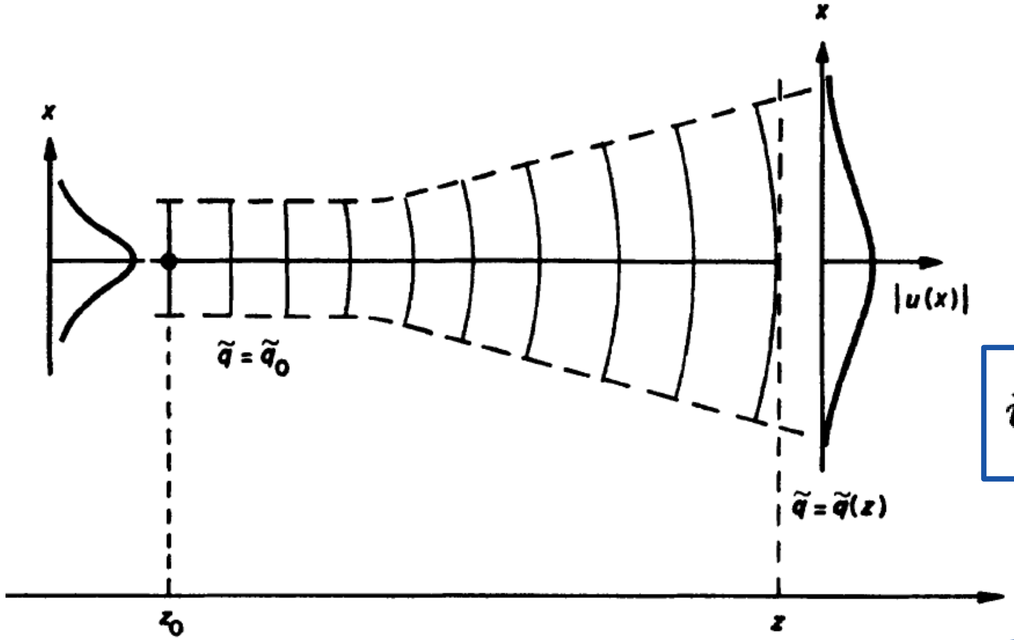
Inherent problem – the wave extends out to infinity in transversal direction!

Gaussian spherical waves: Complex point source

The solution – to introduce a *complex* point source

$x_0 \to 0$;
$y_0 \to 0$; q_0 - complex
$z_0 \to z_0 - q_0$



Substitute radius of curvature R(z) by complex radius

$$\tilde{q}(z) = \tilde{q}_0 + z - z_0$$

Then

$$\tilde{u}(x, y, z) = \frac{1}{\tilde{q}(z)} \exp\left[-jk\frac{x^2 + y^2}{2\tilde{q}(z)}\right]$$

Separate real and imaginary parts of q:

$$\frac{1}{\tilde{q}(z)} \equiv \frac{1}{q_r(z)} - j\,\frac{1}{q_i(z)}$$

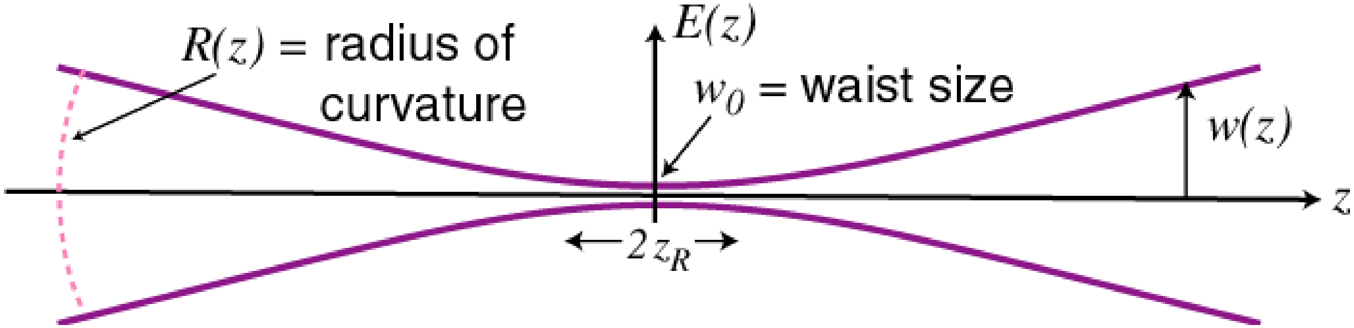
$$\tilde{u}(x, y, z) = \frac{1}{\tilde{q}(z)} \exp\left[-jk\frac{x^2 + y^2}{2q_r(z)} - k\frac{x^2 + y^2}{2q_i(z)}\right]$$

Gaussian spherical waves

Convert into standard notation by denoting: $\frac{1}{\tilde{q}(z)} \equiv \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}$

$$\tilde{u}(x,y,z) = \frac{1}{\tilde{q}(z)} \exp\left[-jk\frac{x^2+y^2}{2R(z)} - \frac{x^2+y^2}{w^2(z)}\right]$$

the lowest-order spherical-gaussian beam solution in free space



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the lowest-order spherical-gaussian beam solution in free space

, where R(z) – the radius of wave front curvature
w(z) – “gaussian spot size”

Note, that R(z) now should be derived from , while $\tilde{q}(z) = \tilde{q}_0 + z - z_0$

The complex source point derivation used is only one of 4 different ways

Gaussian spherical waves: differential approach

From Paraxial Wave Equation approach: $$\frac{\partial \tilde{u}(\mathbf{s}, z)}{\partial z} = -\frac{j}{2k}\nabla_t^2 \tilde{u}(\mathbf{s}, z)$$

Assume a trial solution

$$\tilde{u}(x, y, z) = A(z) \times \exp\left[-jk\frac{x^2 + y^2}{2\tilde{q}(z)}\right]$$

, with A(z) and q(z) being unknown functions

$$\left[\left(\frac{k}{2}\right)^2 \left(\frac{d\tilde{q}}{dz} - 1\right)(x^2 + y^2) - \frac{2jk}{\tilde{q}}\left(\frac{\tilde{q}}{A}\frac{dA}{dz} + 1\right)\right] A(z) = 0$$

$$\frac{d\tilde{q}(z)}{dz} = 1 \qquad \frac{dA(z)}{dz} = -\frac{A(z)}{\tilde{q}(z)}$$

$$\tilde{q}(z) = \tilde{q}_0 + z - z_0 \qquad \frac{A(z)}{A_0} = \frac{\tilde{q}_0}{\tilde{q}(z)}$$

Gaussian spherical waves: differential approach

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$$\tilde{u}(x,y,z) = A(z) \times \exp\left[-jk\frac{x^2+y^2}{2\tilde{q}(z)}\right]$$

$$\tilde{q}(z) = \tilde{q}_0 + z - z_0 \qquad \frac{A(z)}{A_0} = \frac{\tilde{q}_0}{\tilde{q}(z)}$$

$$\tilde{u}(x,y,z) = \frac{A_0\tilde{q}_0}{\tilde{q}(z)}\exp\left[-jk\frac{x^2+y^2}{2\tilde{q}(z)}\right]$$

Leads to the exactly the same solution for the lowest-order spherical-gaussian beam

Higher-Order Gaussian Modes #1

Let's again use a trial solution approach and restrict the problem to the 1D case

$$\tilde{u}_{nm}(x,y,z) = \tilde{u}_n(x,z) \times \tilde{u}_m(y,z)$$

$$\frac{\partial^2 \tilde{u}_n(x,z)}{\partial x^2} - 2jk \frac{\partial \tilde{u}_n(x,z)}{\partial z} = 0.$$

the paraxial wave equation in 1D

Trial solution:

$$\tilde{u}_n(x,z) = A(\tilde{q}(z)) \times h_n\left(\frac{x}{\tilde{p}(z)}\right) \times \exp\left[-jk\frac{x^2}{2\tilde{q}(z)}\right]$$

Considering the propagation rule $d\tilde{q}/dz = 1$

$$h_n'' - 2jk\left[\frac{\tilde{p}}{\tilde{q}} - \tilde{p}'\right] x h_n' - \frac{jk\tilde{p}^2}{\tilde{q}}\left[1 + \frac{2\tilde{q}}{A}\frac{dA}{d\tilde{q}}\right] h_n = 0$$

$q = q(z)$ $\quad$ $p = p(z)$ $\quad$ $h = h\left(\frac{x}{p(z)}\right)$

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$$\tilde{u}_n(x,z) = A(\tilde{q}(z)) \times h_n\left(\frac{x}{\tilde{p}(z)}\right) \times \exp\left[-jk\frac{x^2}{2\tilde{q}(z)}\right]$$

Considering the propagation rule $d\tilde{q}/dz = 1$

$$h_n'' - 2jk\left[\frac{\tilde{p}}{\tilde{q}} - \tilde{p}'\right] x h_n' - \frac{jk\tilde{p}^2}{\tilde{q}}\left[1 + \frac{2\tilde{q}}{A}\frac{dA}{d\tilde{q}}\right] h_n = 0$$

$$H_n'' - 2(x/\tilde{p})H_n' + 2nH_n = 0$$

differential equation for the Hermite polynomials

Higher-Order Gaussian Modes #1

$$h_n'' - 2jk\left[\frac{\tilde{p}}{\tilde{q}} - \tilde{p}'\right] x h_n' - \frac{jk\tilde{p}^2}{\tilde{q}}\left[1 + \frac{2\tilde{q}}{A}\frac{dA}{d\tilde{q}}\right] h_n = 0$$

$$H_n'' - 2(x/\tilde{p})H_n' + 2nH_n = 0$$

$$\frac{d\tilde{p}}{dz} = \frac{\tilde{p}}{\tilde{q}} + \frac{j}{k\tilde{p}}$$

$$\frac{2q}{A}\frac{dA}{d\tilde{q}} = \frac{2jnk\tilde{p}^2}{\tilde{q}} - 1$$

- defines different families of solutions

The "Standard" Hermite Polynomial Solutions

Main assumption $$\frac{1}{\tilde{p}(z)} \equiv \frac{\sqrt{2}}{w(z)}$$

Motivation: solutions with the same normalized shape at every transverse plane z

$$\tilde{u}_n(x,z) = h_n\left(\frac{\sqrt{2}x}{w(z)}\right)\exp\left[\frac{-jkx^2}{2R(z)} - \frac{x^2}{w^2(z)}\right]$$

After proper normalization, one gets expression for **the set of higher-order Hermite-Gaussian mode functions for a beam propagating in free space**

$$\tilde{u}_n(x,z) = \left(\frac{2}{\pi}\right)^{1/4}\left(\frac{1}{2^n n! w_0}\right)^{1/2}\left(\frac{\tilde{q}_0}{\tilde{q}(z)}\right)^{1/2}\left[\frac{\tilde{q}_0}{\tilde{q}_0^*}\frac{\tilde{q}^*(z)}{\tilde{q}(z)}\right]^{n/2}$$

$$\times H_n\left(\frac{\sqrt{2}x}{w(z)}\right)\exp\left[-j\frac{kx^2}{2\tilde{q}(z)}\right],$$

The "Standard" Hermite Polynomial Solutions

$$\tilde{u}_n(x,z) = \left(\frac{2}{\pi}\right)^{1/4} \left(\frac{1}{2^n n! w_0}\right)^{1/2} \left(\frac{\tilde{q}_0}{\tilde{q}(z)}\right)^{1/2} \left[\frac{\tilde{q}_0}{\tilde{q}_0^*}\frac{\tilde{q}^*(z)}{\tilde{q}(z)}\right]^{n/2} H_n\left(\frac{\sqrt{2}x}{w(z)}\right) \exp\left[-j\frac{kx^2}{2\tilde{q}(z)}\right]$$

Rewrite involving the real spot size w(z) and a phase angle ψ(z)

$$\frac{j}{\tilde{q}} = \frac{\lambda}{\pi w^2}\left[1 + j\frac{\pi w^2}{R\lambda}\right] \equiv \frac{\exp[j\psi(z)]}{|\tilde{q}|}$$

reason for the choice: ψ(z)=0 at the waist w(z)=w0

$$\tan\psi(z) \equiv \frac{\pi w^2(z)}{R(z)\,\lambda}$$

"After some algebra":

$$\tilde{u}_n(x) = \left(\frac{2}{\pi}\right)^{1/4} \sqrt{\frac{\exp[-j(2n+1)(\psi(z)-\psi_0)]}{2^n n! w(z)}}\, H_n\left(\frac{\sqrt{2}x}{w(z)}\right) \exp\left[-j\frac{kx^2}{2R(z)} - \frac{x^2}{w^2(z)}\right]$$

And the lowest order gaussian beam mode:

$$\tilde{u}_0(x,z) = \left(\frac{2}{\pi}\right)^{1/4} \sqrt{\frac{\exp j\,[\psi(z)-\psi_0]}{w(z)}}\, \exp\left[-j\frac{kx^2}{2\tilde{q}(z)}\right]$$

Guoy phase shift

$$\tilde{u}_n(x,z) = \left(\frac{2}{\pi}\right)^{1/4} \left(\frac{1}{2^n n! w_0}\right)^{1/2} \left(\frac{\tilde{q}_0}{\tilde{q}(z)}\right)^{1/2} \left[\frac{\tilde{q}_0}{\tilde{q}_0^*}\frac{\tilde{q}^*(z)}{\tilde{q}(z)}\right]^{n/2} H_n\left(\frac{\sqrt{2}x}{w(z)}\right) \exp\left[-j\frac{kx^2}{2\tilde{q}(z)}\right]$$

Rewrite involving the real spot size w(z) and a phase angle ψ(z)

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$$\left[\frac{\tilde{q}_0}{\tilde{q}_0^*}\frac{\tilde{q}^*(z)}{\tilde{q}(z)}\right]^{n/2} \equiv \exp\left[jn\left[\psi(z)-\psi_0\right]\right]$$

at n>0 – gives pure phase shift

Only half of the phase shift comes from each transversal coordinate

Properties of the "Standard" Hermite Polynomial Solutions

- Provide a complete basis set of orthogonal functions

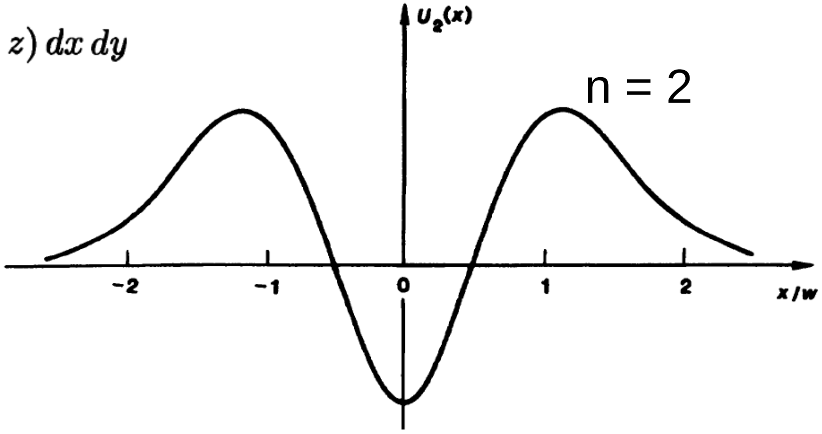
$$\int_{-\infty}^{\infty} u_n^*(x,z)\,\tilde{u}_m(x,z)\,dx = \delta_{nm}$$

$$\tilde{E}(x,y,z) = \sum_n \sum_m c_{nm}\tilde{u}_n(x,z)\tilde{u}_m(y,z)e^{-jkz}$$

arbitrary paraxial optical beam

And expansion coefficients depending on arbitrary choice of w_0 and z_0

$$c_{nm} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{E}(x,y,z)u_n^*(x,z)u_m^*(y,z)\,dx\,dy$$



Properties of the "Standard" Hermite Polynomial Solutions

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$$\int_{-\infty}^{\infty} u_n^*(x,z)\,\tilde{u}_m(x,z)\,dx = \delta_{nm}$$

$$\tilde{E}(x,y,z) = \sum_n \sum_m c_{nm}\tilde{u}_n(x,z)\tilde{u}_m(y,z)e^{-jkz}$$

- Astigmatic modes

$$u_{nm}(x,y,z) = u_n(x,z)\cdot u_m(y,z)$$

q_0 (and w_0,z_0) can have different values in x and y directions of transversal plane   astigmatic Gaussian beam modes

Properties of the "Standard" Hermite Polynomial Solutions

- Provide a complete basis set of orthogonal functions

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q_0 (and w_0,z_0) can have different values in x and y directions of transversal plane astigmatic Gaussian beam modes

- Cylindrical coordinates: Laguerre-Gaussian modes

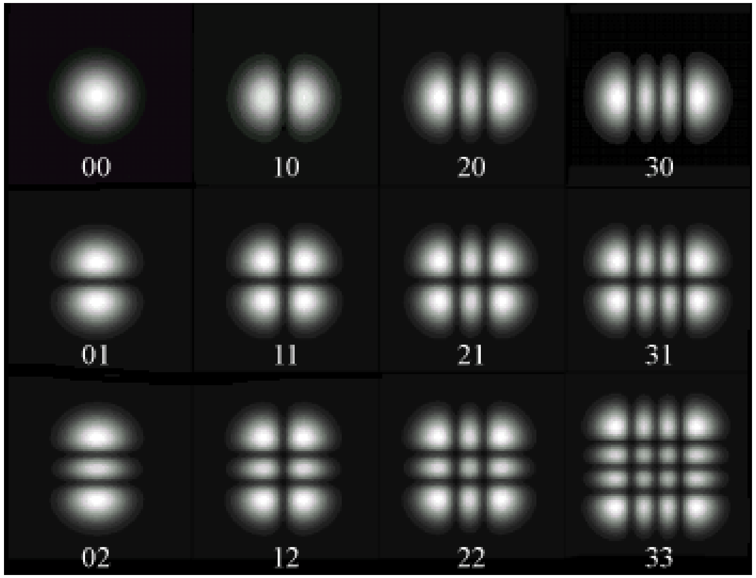
$$\tilde{u}_{pm}(r,\theta,z) = \sqrt{\frac{2p!}{(1+\delta_{0m})\,\pi\,(m+p)!}}\;\frac{\exp j(2p+m+1)(\psi(z)-\psi_0)}{w(z)}\left(\frac{\sqrt{2}r}{w(z)}\right)^m L_p^m\left(\frac{2r^2}{w(z)^2}\right)\exp\left[-jk\frac{r^2}{2\tilde{q}(z)} + im\theta\right]$$

$p \geq 0$ - radial index

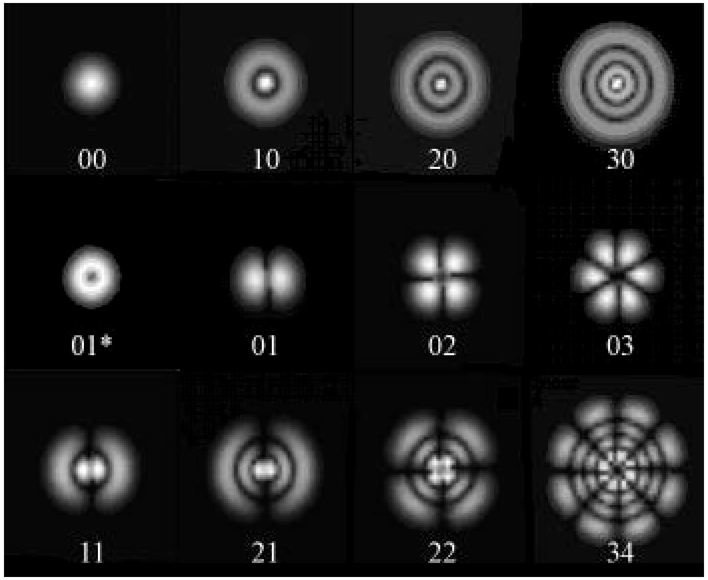
m - asimuthal index

Properties of the "Standard" Hermite Polynomial Solutions

Hermite-Gaussian laser modes



Laguerre-Gaussian laser modes



The "Elegant" Hermite Polynomial Solutions

Main assumption $$\frac{1}{\tilde{p}(z)} \equiv \sqrt{\frac{jk}{2\tilde{q}(z)}}$$

Motivation: having the same complex argument in Hermite ploynomial and gaussian exponent

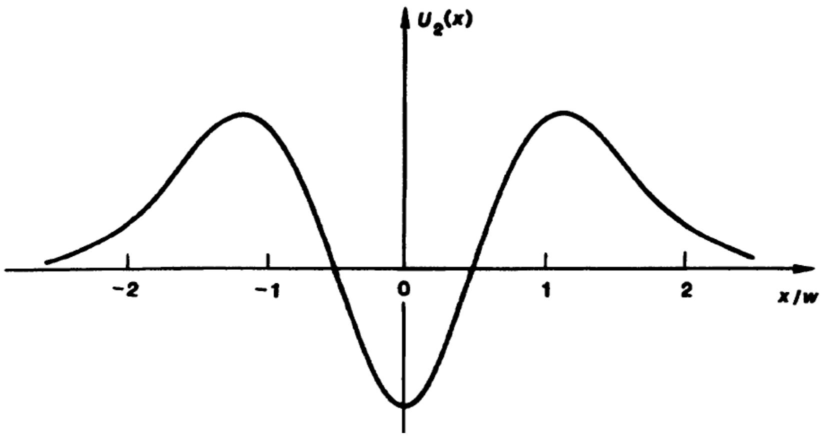
$$\hat{u}_n(x,z) = \hat{u}_0 \left[\frac{\tilde{q}_0}{\tilde{q}(z)}\right]^{n+1/2} H_n\left(\sqrt{\frac{jkx^2}{2\tilde{q}(z)}}\right) \exp\left[-j\frac{kx^2}{2\tilde{q}(z)}\right]$$

- biorthogonal to a set of adjoint functions $\hat{v}_n(x,z) = H_n\left(\sqrt{\frac{-jk}{2\tilde{q}^*}}x\right)$

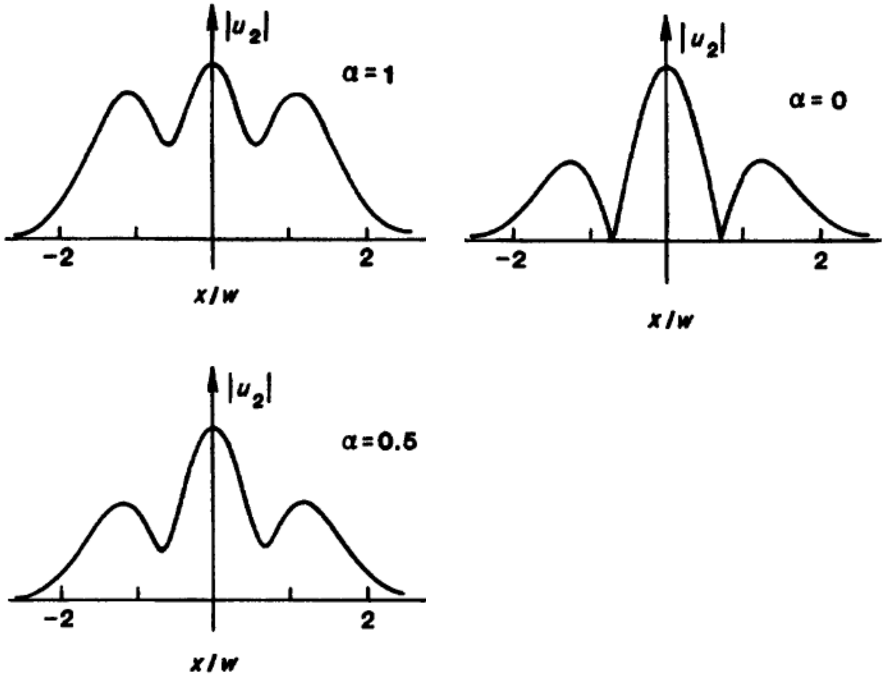
$$\int_{-\infty}^{\infty} \hat{u}_n(x,z)\,\hat{v}_n^*(x,z)\,dx = c_n\delta_{nm}$$

- significant difference in high order modes with “standard” sets

The "standard" and "elegant" sets high-order solutions



$$\tilde{u}_2(x,z) = \text{const} \times \left[\frac{4x^2}{w^2} - 1\right] \exp\left[-j\frac{kx^2}{2\tilde{q}}\right]$$

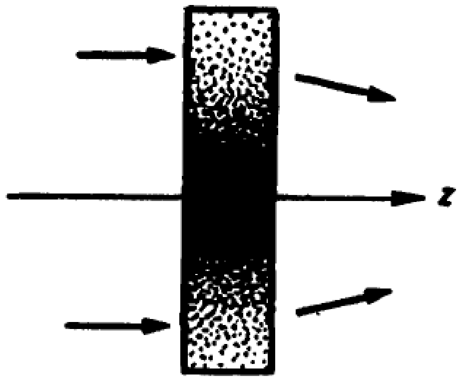


$$\hat{u}_2(x) = \text{const} \times \left[\frac{2(1+j\alpha)x^2}{w^2} - 1\right] \exp\left[-j\frac{kx^2}{2\tilde{q}}\right]$$

, with $\alpha = \dfrac{\pi w^2}{R\lambda}$

Gaussian beam propagation in ducts

Duct – is a graded index optical waveguided



$$n(r) = n_0 - \frac{1}{2} n_2 r^2$$

$$\left[\nabla_{xy}^2 - k^2 n_2 (x^2 + y^2) - 2jk \frac{\partial}{\partial z}\right] \tilde{u}(x, y, z) = 0$$

Solution:

$$\tilde{u}(x, y, z) = \tilde{u}_0 \exp\left[-\frac{x^2 + y^2}{w_1^2} + j \frac{\lambda z}{w_1^2}\right]$$

$$w_1^2 = \frac{\lambda}{\pi \sqrt{n_2}}$$

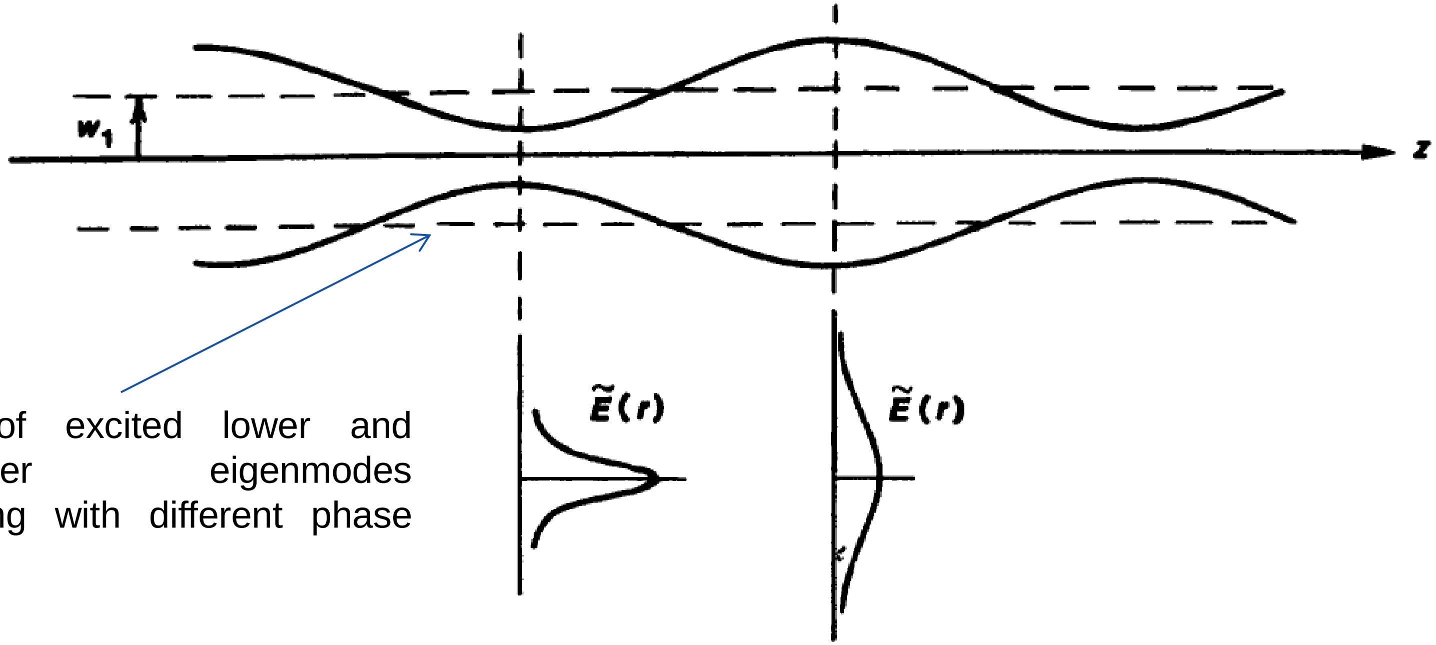
Gaussian eigenmode of the duct

w<< $1/n_2^{1/2}$

Gaussian beam propagation in ducts

Duct – is a graded index optical waveguided $n(r) = n_0 - \frac{1}{2} n_2 r^2$

$$\tilde{u}(x,y,z) = \tilde{u}_0 \exp\left[-\frac{x^2+y^2}{w_1^2} + j\,\frac{\lambda z}{w_1^2}\right]$$



Beating of excited lower and higher-order eigenmodes propagating with different phase velocities

Numerical Beam Propagation Methods

1. Finite Difference Approach

$$\frac{\partial \tilde{u}(s,z)}{\partial z} = -\frac{j}{2k}\nabla_t^2 \tilde{u}(s,z)$$

Beam propagation through inhomogeneous regions

Numerical Beam Propagation Methods

1. Finite Difference Approach

$$\frac{\partial \tilde{u}(s,z)}{\partial z} = -\frac{j}{2k}\nabla_t^2 \tilde{u}(s,z)$$

2. Fourier Transform Interpretation of Huygens Integral

$$\tilde{u}(x,z) = \sqrt{\frac{j}{L\lambda}} \int \tilde{u}_0(x_0, z_0) \exp\left[-j\frac{\pi(x-x_0)^2}{L\lambda}\right] dx_0$$

$$\tilde{u}(x,z) = \tilde{u}_0(x_0) * \exp[-j\pi x_0^2/(z-z_0)\lambda]$$

xN FFT

x1 FFT

$$f * g = \mathcal{F}^{-1}\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\}$$ → remains a Gaussian

Numerical Beam Propagation Methods

3. Alternative Fourier Transform Approach

$$\tilde{u}(x,z) = \exp\left(\frac{-j\pi x^2}{L\lambda}\right)\sqrt{\frac{j}{L\lambda}}\int_{-\infty}^{\infty}\tilde{u}_0'(x_0,z_0)\times\exp[j(2\pi/L\lambda)xx_0]\,dx_0$$

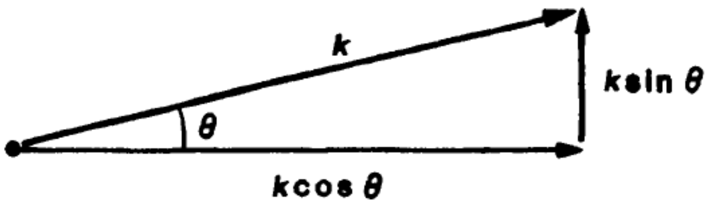
$$\tilde{u}_0'(x_0,z_0) \equiv \tilde{u}_0(x_0,z_0)\,e^{-j\pi x_0^2/L\lambda}$$

the Huygens-Fresnel propagation integral appears as a single (scaled) Fourier transform between the input and output functions u_0 and u

single FT, but applied to a more complex input fucntion

Paraxial Plane Waves and Transverse Spatial Frequencies

FT → expansion of the optical beam in a set of infinite plane waves traveling in slightly different directions



Set of infinite plane waves

$$\tilde{u}_{pw}(x,y,z) \equiv \exp[-j\boldsymbol{k}\cdot\boldsymbol{r}] = \exp\left[-j(k_x x + k_y y + k_z z)\right]$$

$\boldsymbol{k}$ →

$$k_x = k\sin\theta_x \equiv 2\pi s_x$$

$$k_y = k\sin\theta_y \equiv 2\pi s_y$$

$$k_z = k - \pi\lambda(s_x^2 + s_y^2)$$

→ θ_x,θ_y or
spatial frequencies: s_x, s_y

$$\tilde{u}_{pw}(x,y,z) = \tilde{u}_{pw}(x,y,0) \times \exp[-jkz + j\pi\lambda(s_x^2 + s_y^2)z]$$

$$\tilde{u}(x,y,z) = \int\!\!\int \tilde{U}_{pw}(s_x, s_y, z) \times e^{-j2\pi(s_x x + s_y y)}\, ds_x\, ds_y$$

KTH ROYAL INSTITUTE
OF TECHNOLOGY

Physical Properties of Gaussian Beams

Ruslan Ivanov
OFO/ICT

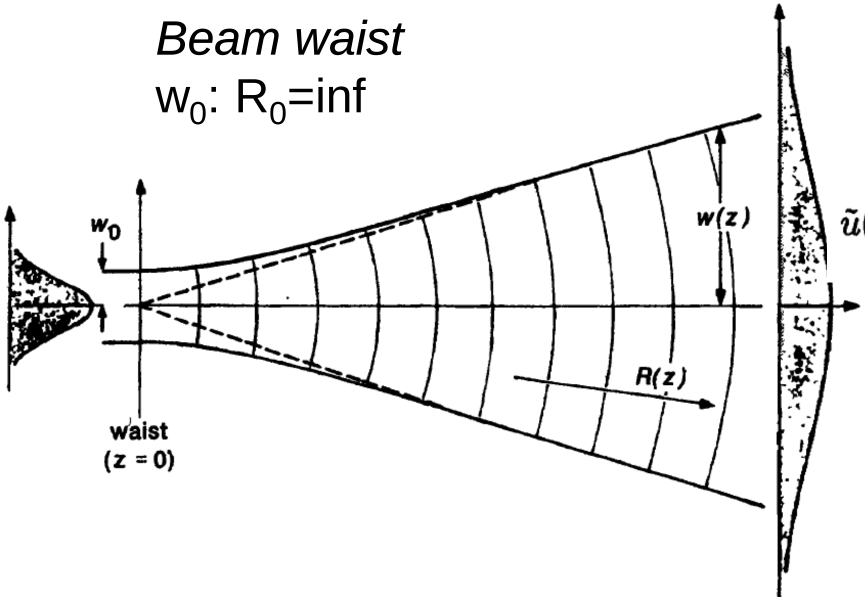
Outline

- Gaussian beam propagation
  - Aperture transmission
  - Beam collimation
  - Wavefront radius of curvature
- Gaussian beam focusing
  - Focus spot sizes and focus depth
  - Focal spot deviation
- Lens law and Gaussian mode matching
- Axial phase shifts
- Higher-order Gaussian modes
  - Hermite-Gaussian patterns
  - Higher-order mode sizes and aperturing
  - Spatial-frequency consideration

Gaussian beam

Beam waist

w_0: R_0=inf



"Standard" hermite-gaussian solution (n=0)

$$\tilde{u}(x,y,z) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\tilde{q}_0}{w_0 \tilde{q}(z)} \exp\left[-jkz - jk\frac{x^2+y^2}{2\tilde{q}(z)}\right]$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \frac{\exp[-jkz + j\psi(z)]}{w(z)} \exp\left[-\frac{x^2+y^2}{w^2(z)} - jk\frac{x^2+y^2}{2R(z)}\right]$$

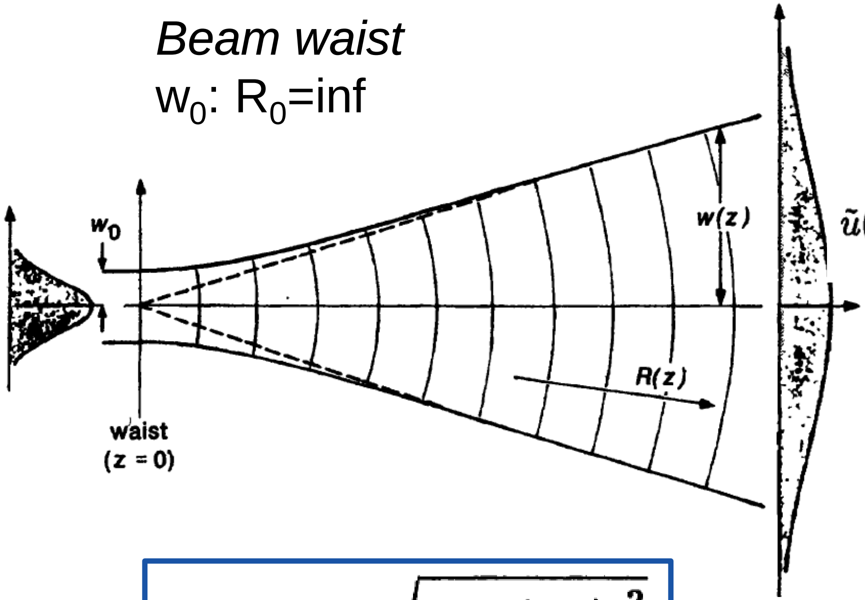
, where

$$\frac{1}{\tilde{q}(z)} \equiv \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}$$

$$\tilde{q}(z) = \tilde{q}_0 + z = z + jz_R \qquad \tilde{q}_0 = j\frac{\pi w_0^2}{\lambda} = jz_R$$

Gaussian beam

Beam waist
w_0: R_0=inf



"Standard" hermite-gaussian solution (n=0)

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$$= \left(\frac{2}{\pi}\right)^{1/2} \frac{\exp[-jkz + j\psi(z)]}{w(z)} \exp\left[-\frac{x^2+y^2}{w^2(z)} - jk\frac{x^2+y^2}{2R(z)}\right]$$

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$$\frac{1}{\tilde{q}(z)} \equiv \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}$$

$$\tilde{q}(z) = \tilde{q}_0 + z = z + jz_R \qquad \tilde{q}_0 = j\frac{\pi w_0^2}{\lambda} = jz_R$$

$$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_R}\right)^2}$$

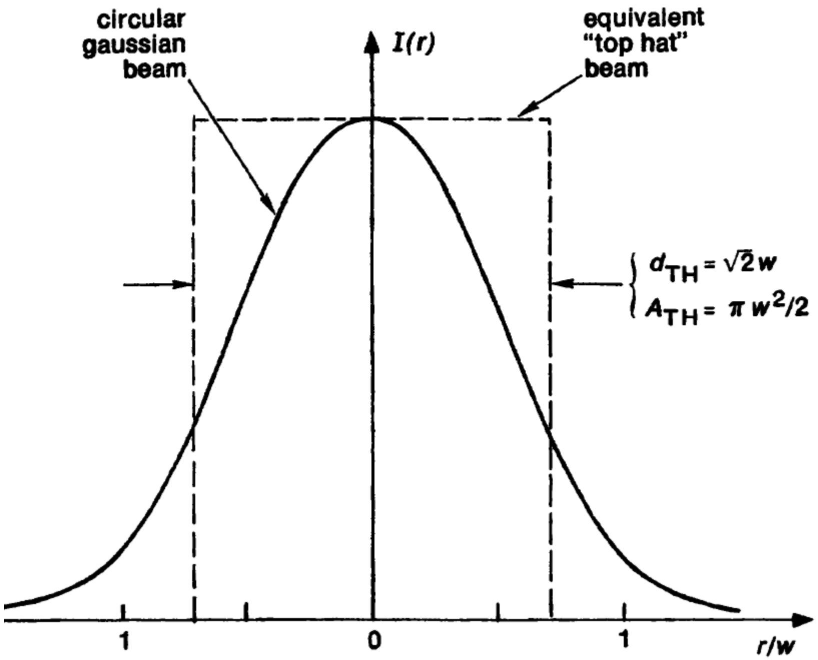
$$R(z) = z + \frac{z_R^2}{z}$$

$$\psi(z) = \tan^{-1}\left(\frac{z}{z_R}\right)$$

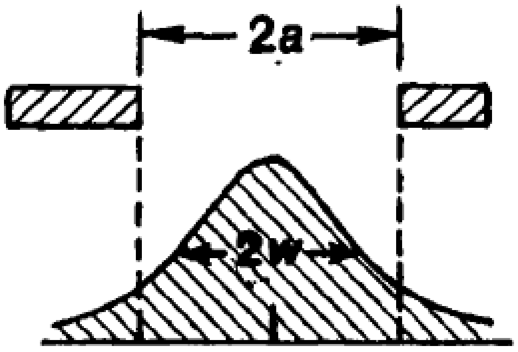
Aperture transmission

The radial intensity variation of the beam

$$I(r) = \frac{2P}{\pi w^2} e^{-2r^2/w^2}$$

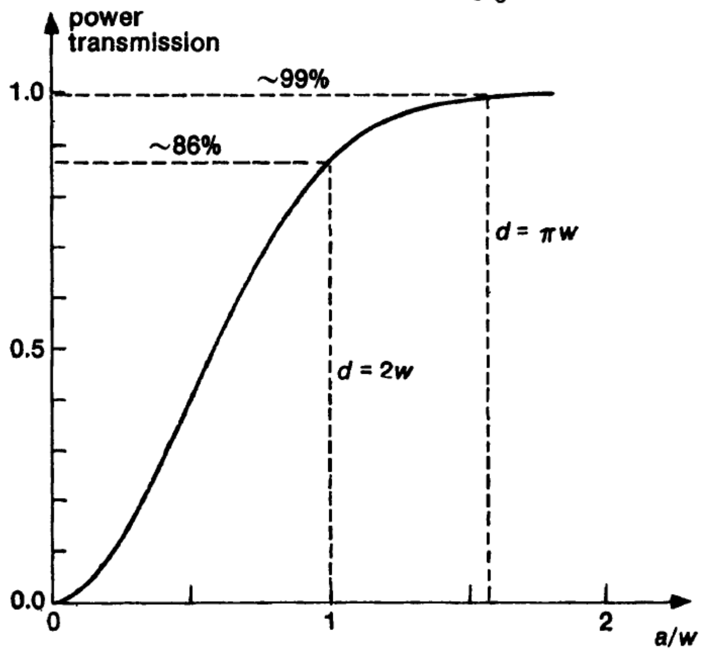


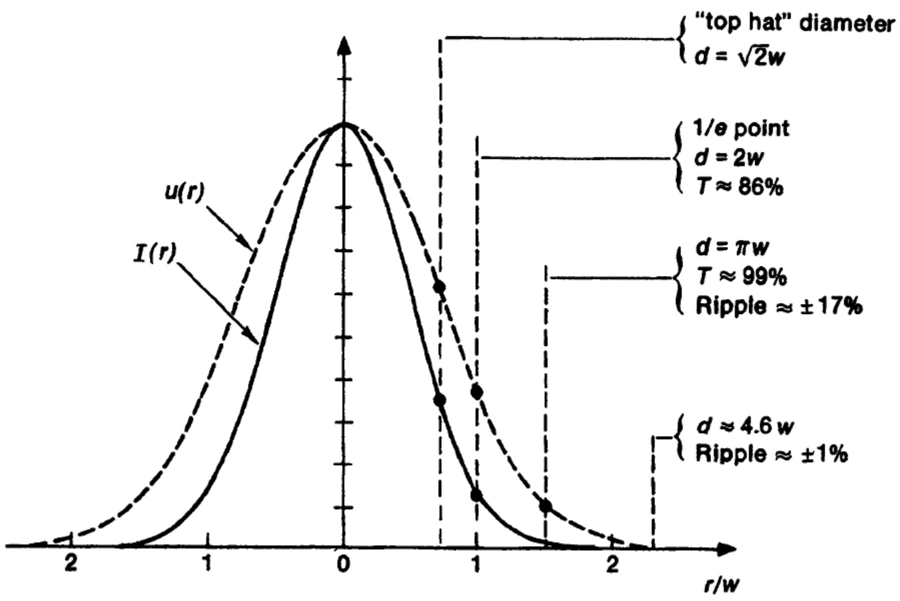
Aperture transmission



The radial intensity variation of the beam

$$\text{power transmission} = \frac{2}{\pi w^2}\int_0^a 2\pi r e^{-2r^2/w^2}\,dr = 1 - e^{-2a^2/w^2}$$





Aperture transmission

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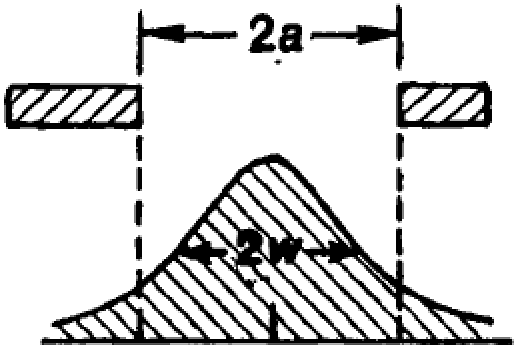
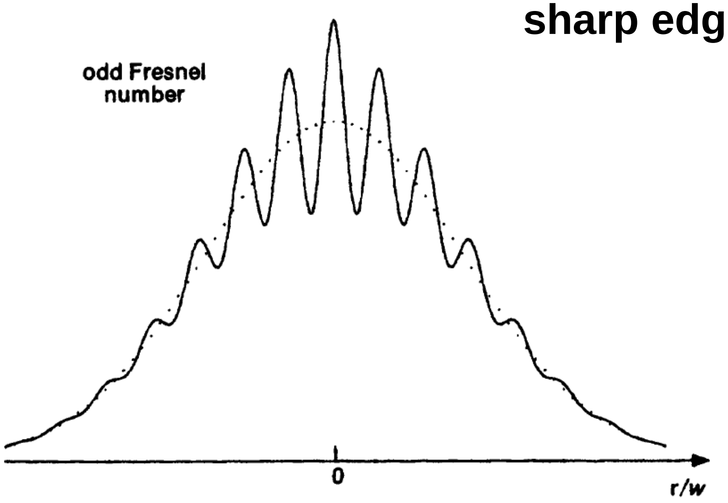
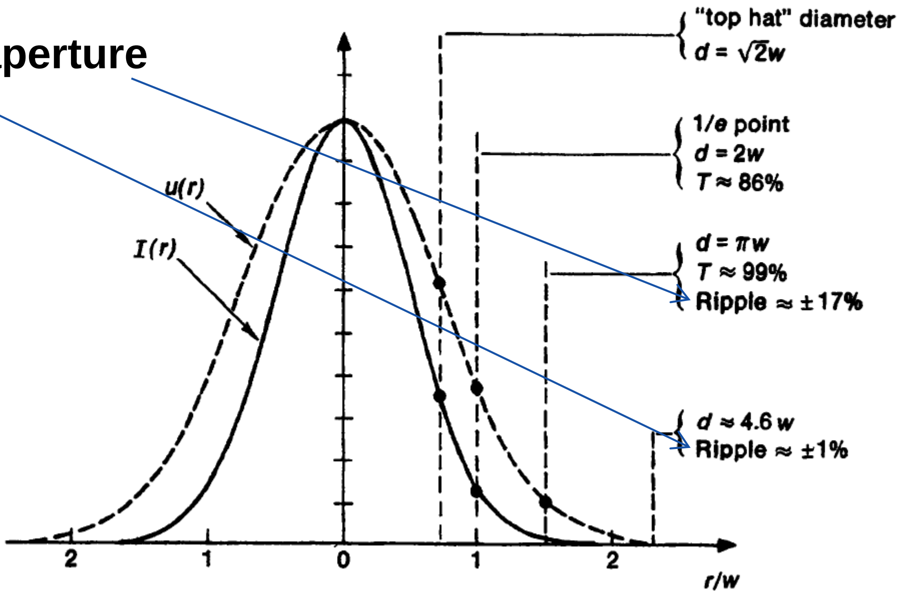
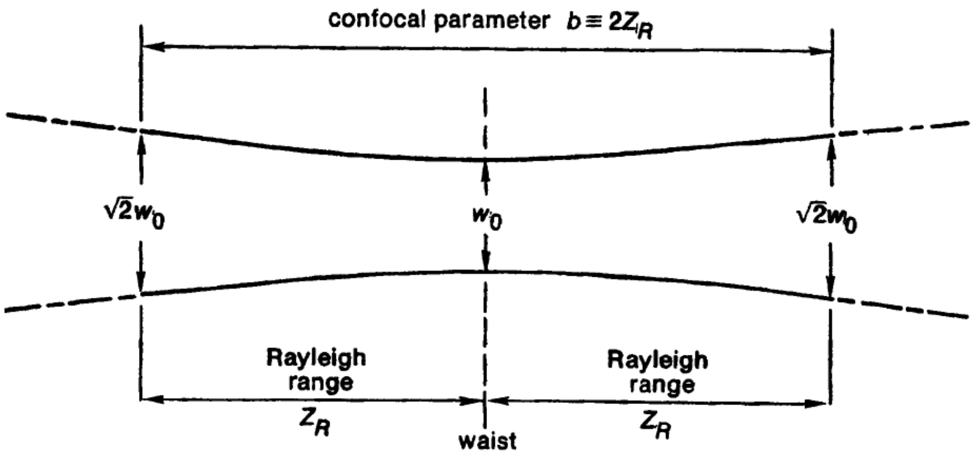


+ diffraction on aperture sharp edges



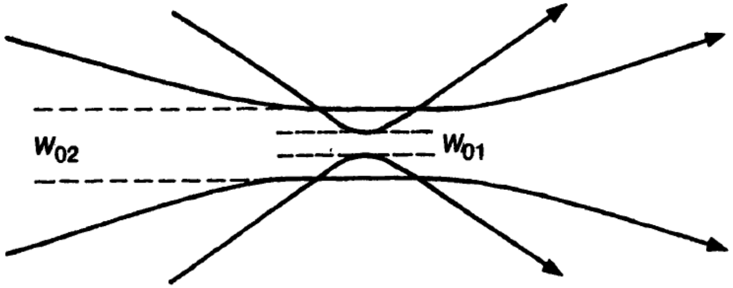


Gaussian beam collimation



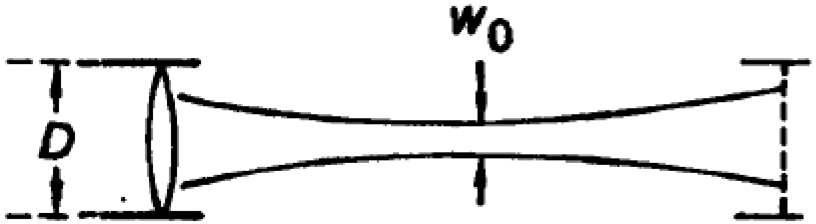
$$w(z_R) = \sqrt{2}w_0$$

$$z = z_R \equiv \frac{\pi w_0^2}{\lambda} = \text{“Rayleigh range.”}$$



z_R characterizes switch from near-field (collimated beam) to far-field (linearly divergent beam)

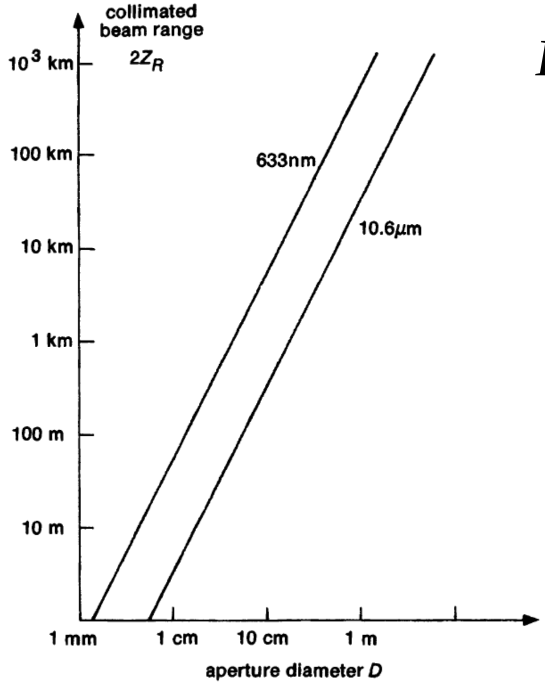
Collimated Gaussian beam propagation



$$z = z_R \equiv \frac{\pi w_0^2}{\lambda} = \text{"Rayleigh range."}$$

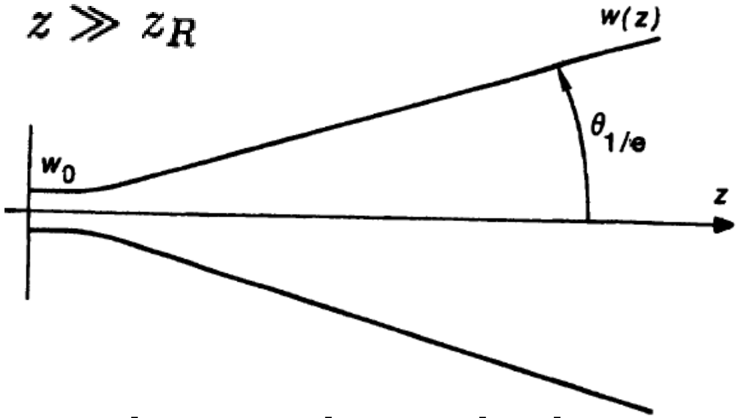
$$\text{collimated range} = 2z_R = \frac{2\pi w_0^2}{\lambda} \approx \frac{D^2}{\pi\lambda}$$

$D = \pi\sqrt{2}w_0$ (99% criterion)



Far-field Gaussian beam propagation

$z \gg z_R$



$$z = z_R \equiv \frac{\pi w_0^2}{\lambda} = \text{“Rayleigh range.”}$$

$$w(z) \approx \frac{w_0 z}{z_R} = \frac{\lambda z}{\pi w_0}$$

$$\boxed{w_0 \times w(z) \approx \frac{\lambda z}{\pi}}$$

1. The “Top-hat” criterion

$$I_{\text{axis}}(z) = \frac{2P}{\pi w^2(z)} \approx \frac{P}{\lambda^2 z^2 / 2\pi w_0^2}$$

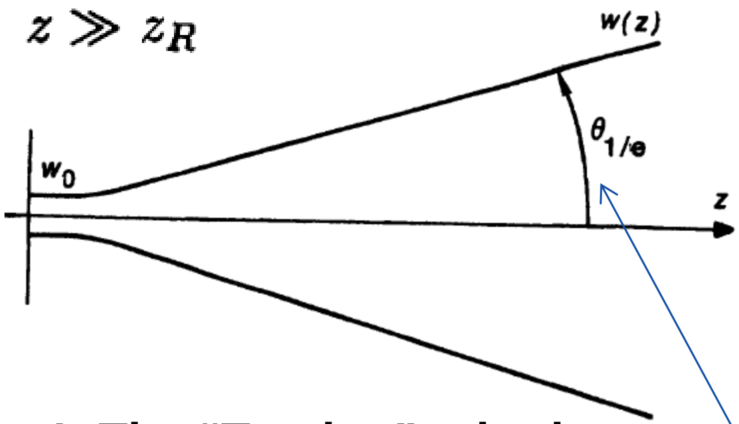
$$\Omega_{\text{TH}} = \frac{\pi w^2(z)}{2z^2} = \frac{\lambda^2}{2\pi w_0^2}$$

$$A_{TH} = \frac{\pi w_0^2}{2}$$ - effective source aperture area

$$A_{\text{TH}} \times \Omega_{\text{TH}} = \left(\frac{\lambda}{2}\right)^2$$

Far-field Gaussian beam propagation

$z \gg z_R$



$$z = z_R \equiv \frac{\pi w_0^2}{\lambda} = \text{“Rayleigh range.”}$$

$$w(z) \approx \frac{w_0 z}{z_R} = \frac{\lambda z}{\pi w_0}$$

$$\boxed{w_0 \times w(z) \approx \frac{\lambda z}{\pi}}$$

1. The “Top-hat” criterion

$$A_{\mathrm{TH}} \times \Omega_{\mathrm{TH}} = \left(\frac{\lambda}{2}\right)^2$$

2. The 1/e criterion

$$A_{1/e} \equiv \pi w_0^2$$

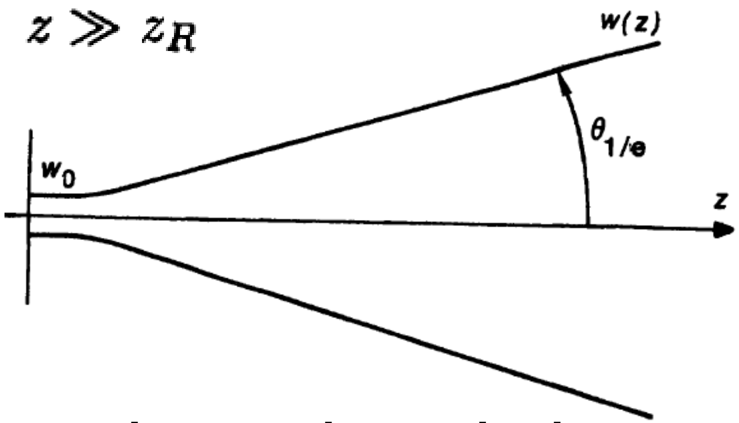
$$\theta_{1/e} = \lim_{z \to \infty} \frac{w(z)}{z} = \frac{\lambda}{\pi w_0}$$

$$\Omega_{1/e} = \pi \theta_{1/e}^2 = \frac{\lambda^2}{\pi w_0^2}$$

$$A_{1/e}\Omega_{1/e} = \pi w_0^2 \times \pi \theta_{1/e}^2 = \lambda^2$$

$$\boxed{\iint A(\Omega)\, d\Omega = \lambda^2}$$ - Antenna theorem

Far-field Gaussian beam propagation



$$z = z_R \equiv \frac{\pi w_0^2}{\lambda} = \text{“Rayleigh range.”}$$

$$w(z) \approx \frac{w_0 z}{z_R} = \frac{\lambda z}{\pi w_0}$$

$$w_0 \times w(z) \approx \frac{\lambda z}{\pi}$$

1. The “Top-hat” criterion

$$A_{\mathrm{TH}} \times \Omega_{\mathrm{TH}} = \left(\frac{\lambda}{2}\right)^2$$

2. The 1/e criterion

$$A_{1/e}\Omega_{1/e} = \pi w_0^2 \times \pi\theta_{1/e}^2 = \lambda^2$$

3. The conservative criterion

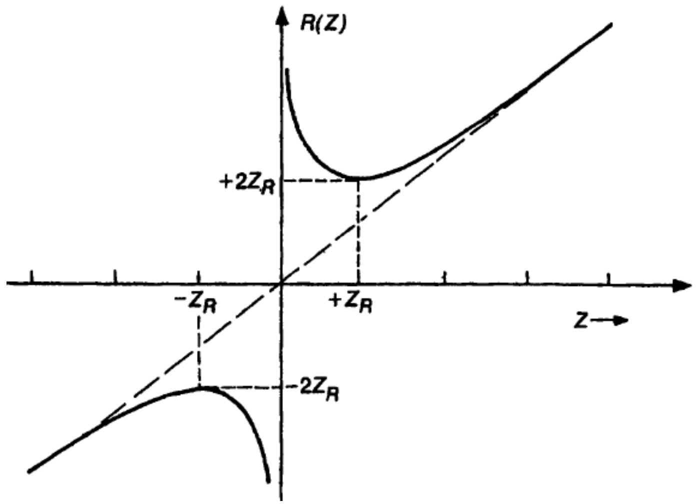
far-field beam angle

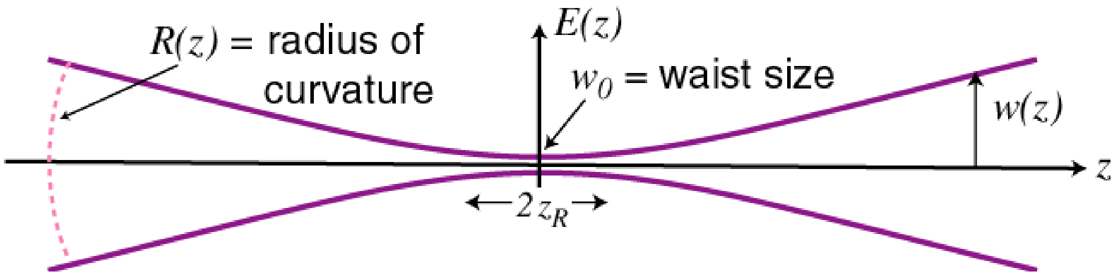
$$A_\pi\Omega_\pi = \left(\frac{\pi}{2}\right)^4 \lambda^2 \approx 6\lambda^2$$

Far-field Gaussian beam propagation

Wavefront radius of curvature

$$R(z) = z + \frac{z_R^2}{z} \approx \begin{cases} \infty & \text{for} \quad z \ll z_R \\ 2z_R & \text{for} \quad z = z_R \\ z & \text{for} \quad z \gg z_R \end{cases}$$

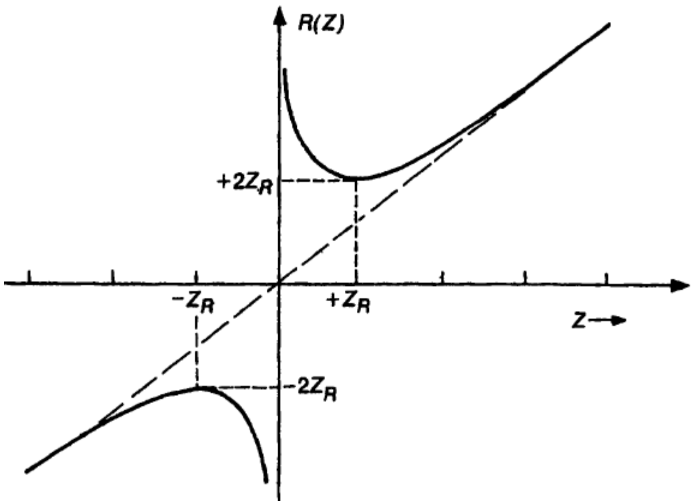




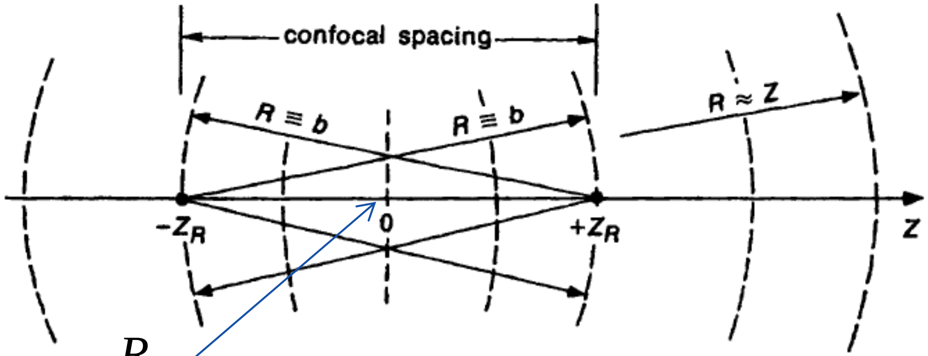
Far-field Gaussian beam propagation

Wavefront radius of curvature

$$R(z) = z + \frac{z_R^2}{z} \approx \begin{cases} \infty & \text{for} \quad z \ll z_R \\ 2z_R & \text{for} \quad z = z_R \\ z & \text{for} \quad z \gg z_R \end{cases}$$



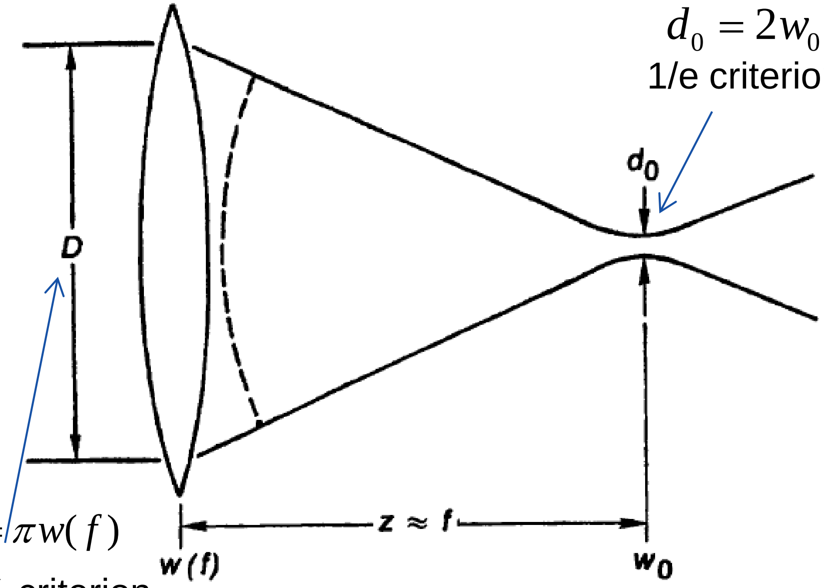
Put two curved mirrors of radius R at the points $\pm z_R$ to match exactly the wavefronts R(z)



$$f = \frac{R}{2}$$

- Symmetric confocal resonator

Gaussian beam focusing



$d_0 = 2w_0$

1/e criterion

$D = \pi w(f)$

99% criterion

$$w_0 \times w(f) \approx \frac{f\lambda}{\pi}$$

Lens is in the far-field

1. Focused spot size

$$d_0 \approx \frac{2f\lambda}{D}$$

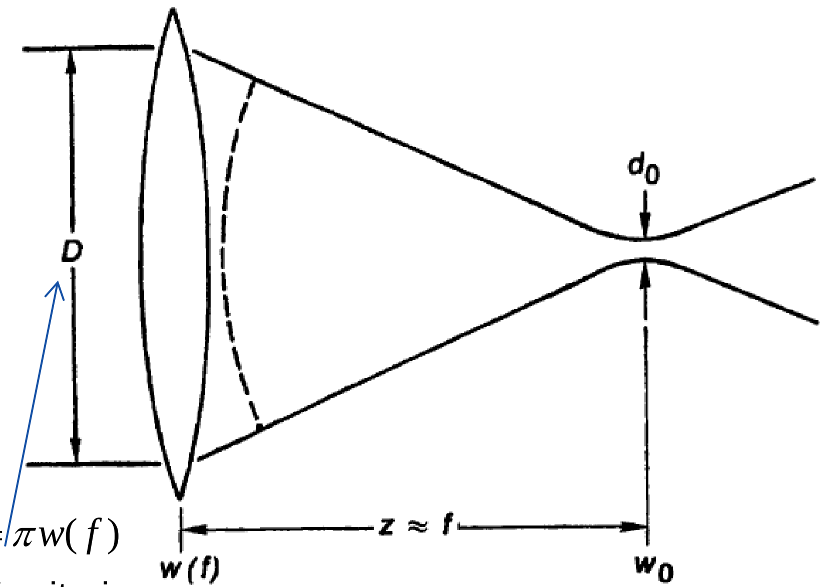
lens radius *a*

$$f^\# \equiv \frac{f}{D} \qquad d_0 \approx 2f^\#\lambda$$

$$N_f \equiv \frac{a^2}{f\lambda} \qquad \frac{d_0}{D} \approx \frac{1}{2N_f}$$

Larger gaussian beam is required for *stronger* focusing

Gaussian beam focusing



$D = \pi w(f)$

99% criterion

$$w_0 \times w(f) \approx \frac{f\lambda}{\pi}$$

1. Focused spot size

$$d_0 \approx \frac{2f\lambda}{D}$$

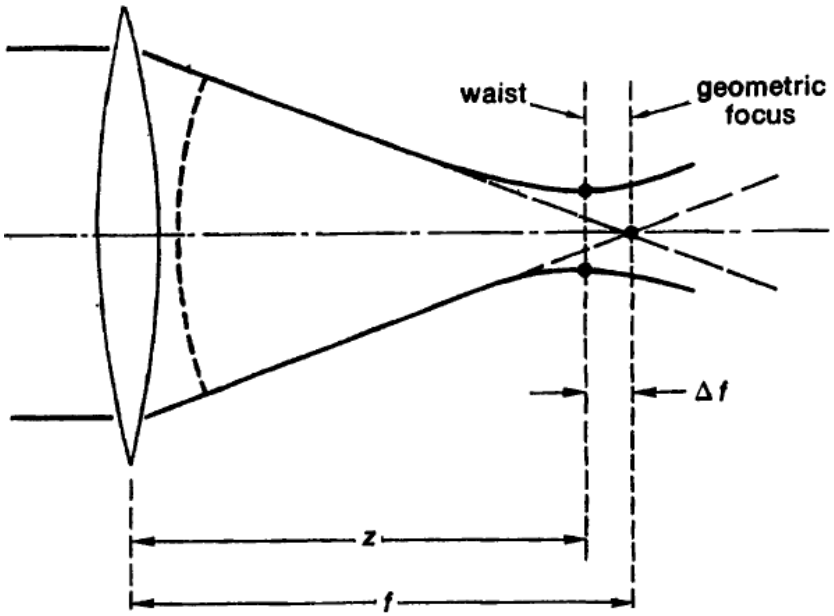
2. Depth of focus

- Region in which the beam can be thought collimated

$$\text{depth of focus } = 2z_R \approx 2\pi f^{\#2}\lambda \approx \frac{\pi}{2}\left(\frac{d_0}{\lambda}\right)^2 \lambda$$

The beam focused to a spot Nλ in diameter will be $N^2\lambda$ in length

Gaussian beam focusing



$$w_0 \times w(f) \approx \frac{f\lambda}{\pi}$$

1. Focused spot size

$$d_0 \approx \frac{2f\lambda}{D}$$

2. Depth of focus

$$\text{depth of focus } = 2z_R \approx 2\pi f^{\#^2}\lambda \approx \frac{\pi}{2}\left(\frac{d_0}{\lambda}\right)^2\lambda$$

3. Focal spot deviation

$$R(z) = z + z_R^2/z = f$$

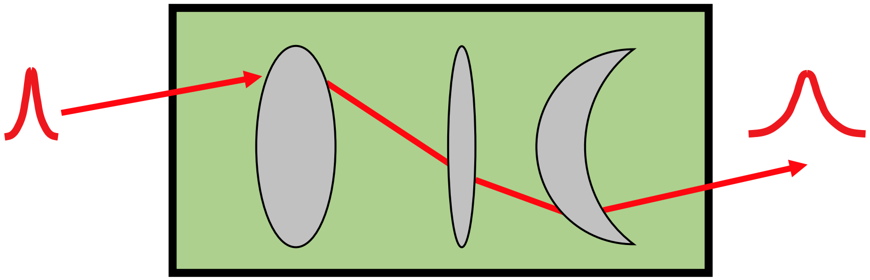
$$\Delta f \equiv f - z = z_R^2/z \approx z_R^2/f$$

$$\frac{\Delta f}{f} \approx \frac{1}{2N_f^2}$$

$\Delta f << depth\_of\_focus$ - The effect is usually negligible (z_R<<f)

Gaussian Mode Matching

The problem: convert w_1 at z_1 to w_2 at z_2



Thin lens law

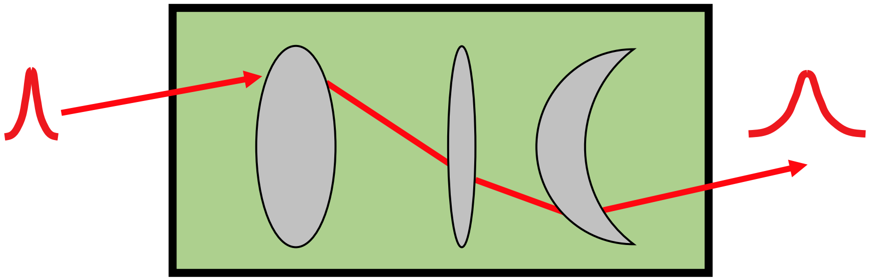
$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$$

The lens law for gaussian beams

$$\frac{1}{\tilde{q}_2} = \frac{1}{\tilde{q}_1} - \frac{1}{f}$$

Gaussian Mode Matching

The problem: convert w_1 at z_1 to w_2 at z_2

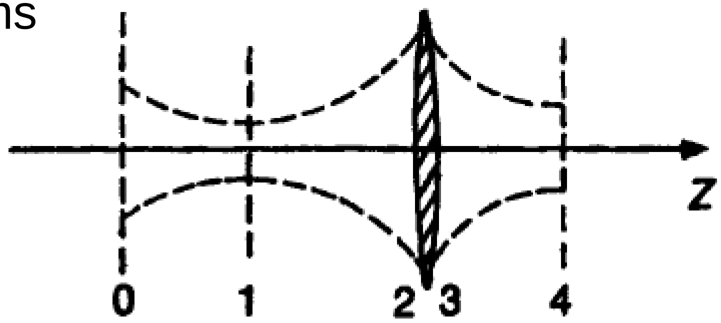


Thin lens law:

$$\frac{1}{R_2} = \frac{1}{R_1} - \frac{1}{f}$$

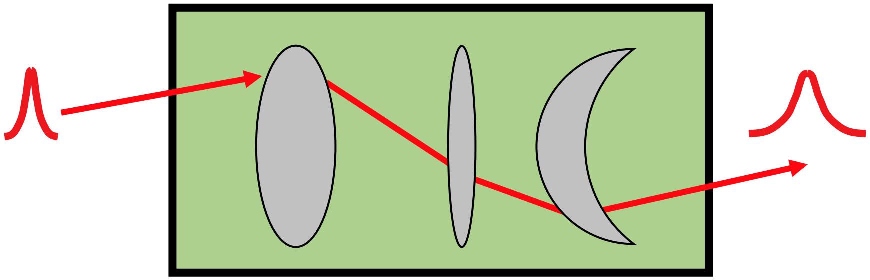
The lens law for gaussian beams

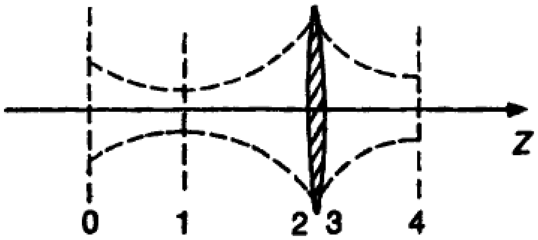
$$\begin{cases} \dfrac{1}{\tilde{q}_2} = \dfrac{1}{\tilde{q}_1} - \dfrac{1}{f} \\ \tilde{q}_2 = \tilde{q}_1 + z_2 - z_1 \end{cases} \Rightarrow$$



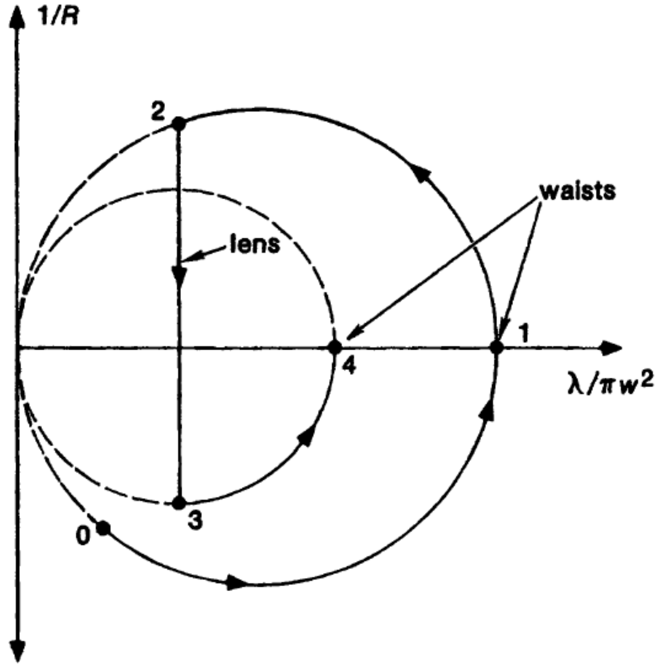
Gaussian Mode Matching

The problem: convert w_1 at z_1 to w_2 at z_2





Gaussian-beam (Collins) chart



The lens law for gaussian beams

$$\frac{1}{\tilde{q}_2} = \frac{1}{\tilde{q}_1} - \frac{1}{f}$$

$$\frac{1}{\tilde{q}(z)} \equiv \frac{1}{R(z)} - j\frac{\lambda}{\pi w^2(z)}$$

Axial phase shifts

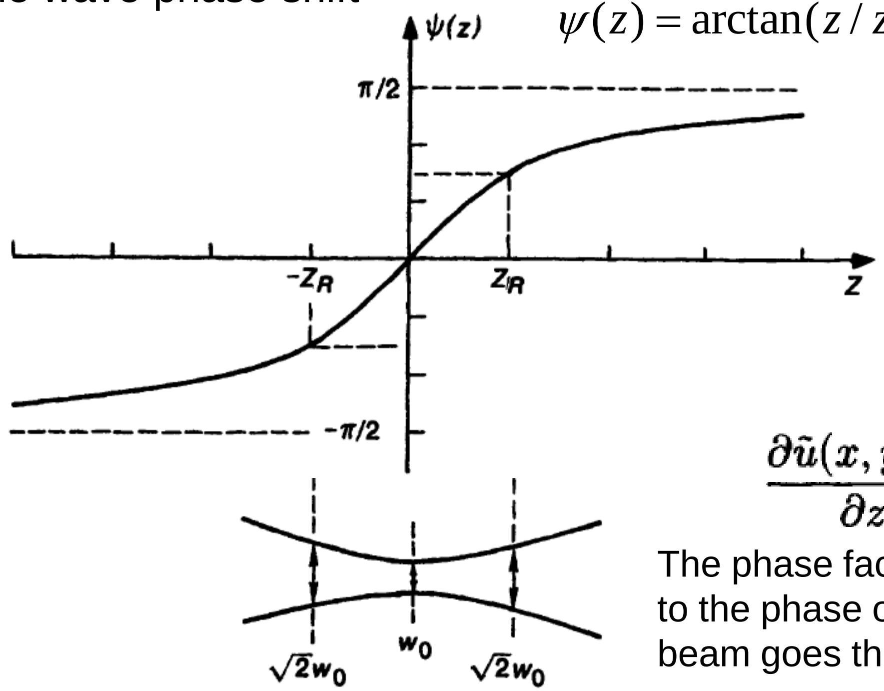
Cumulative phase shift variation on the optical axis:

$$\bar{u}(z) \propto \frac{\tilde{q}_0 e^{-jkz}}{\tilde{q}(z)} = \frac{e^{-jkz}}{1 - jz/z_R} = \frac{\exp[-jkz + j\psi(z)]}{w(z)}$$

Plane wave phase shift

Added phase shift

$$\psi(z) = \arctan(z / z_R) \rightarrow \begin{cases} \pi / 2 \text{ when } z \rightarrow +\infty \\ -\pi / 2 \text{ when } z \rightarrow -\infty \end{cases}$$

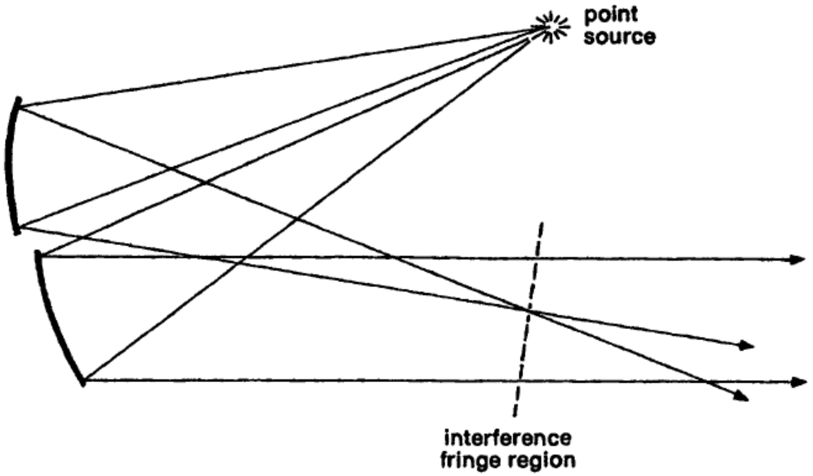


$$\frac{\partial \tilde{u}(x,y,z)}{\partial z} = -\frac{j}{2k} \nabla^2_{xy} \tilde{u}(x,y,z)$$

The phase factor yields a phase shift relative to the phase of a plane wave when a Gaussian beam goes through a focus.

Axial phase shifts: The Guoy effect

Valid for the beams with any reasonably simple cross section



More pronounced for the higher modes:

$$(n+m+1)\times\psi(z)$$

1D→ $(n + 1/2)\psi(z)$

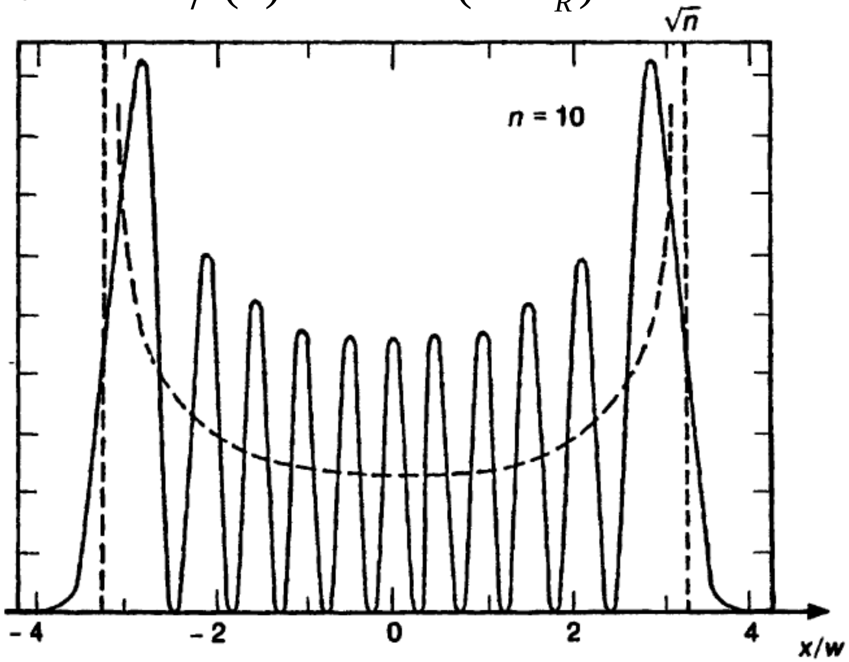
Each wavelet will acquire exactly π/2 of extra phase shift in diverging from its point source or focus to the far field

Higher-Order Gaussian Modes

Hermite-Gaussian TEM_{nm}

$$\tilde{u}_n(x,z) = \left(\frac{2}{\pi}\right)^{1/4} \left(\frac{\exp[j(2n+1)\psi(z)]}{2^n n! w(z)}\right)^{1/2} H_n\left(\frac{\sqrt{2}x}{w(z)}\right) \exp\left[-jkz - j\frac{kx^2}{2R(z)} - \frac{x^2}{w^2(z)}\right]$$

, where $\psi(z) = \arctan(z / z_R)$



Higher-Order Gaussian Modes

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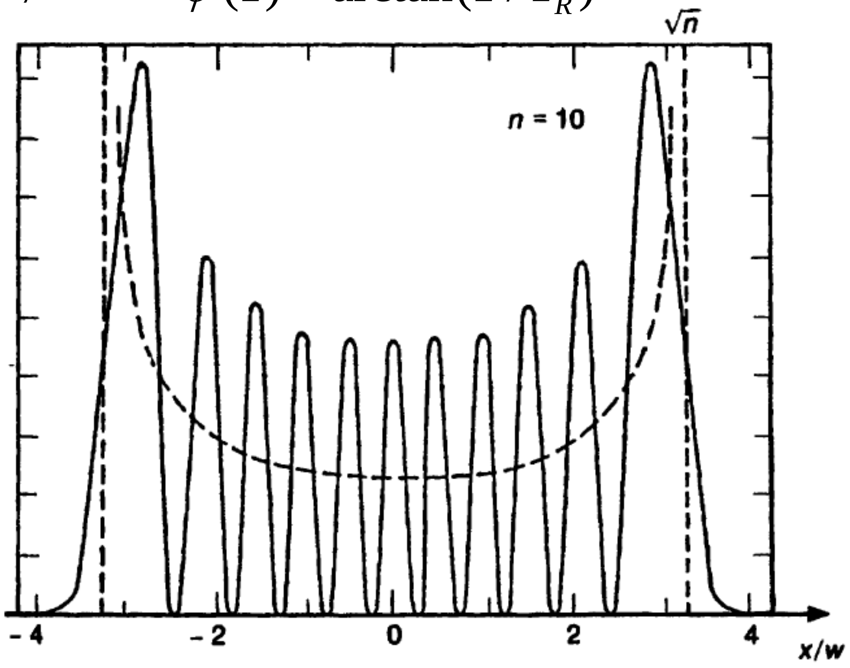
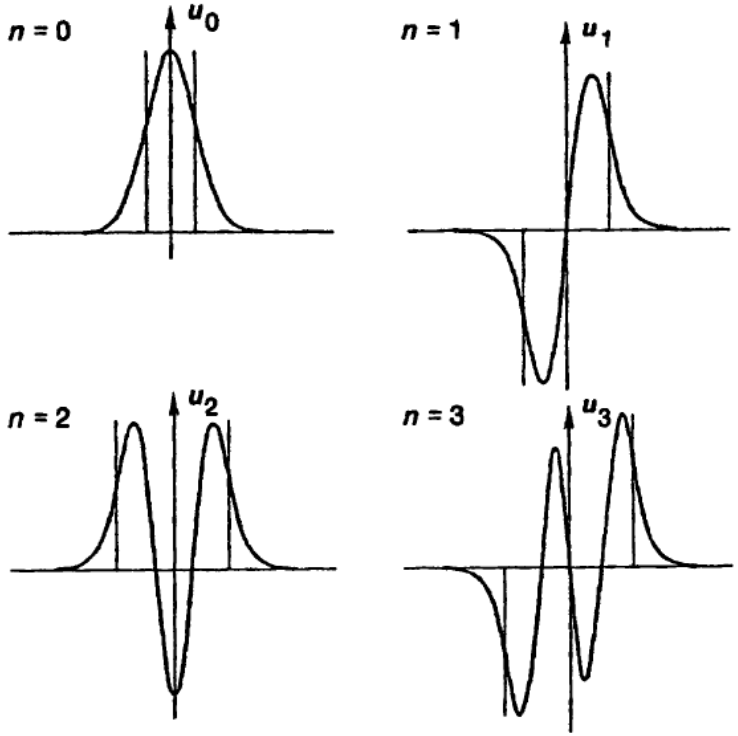
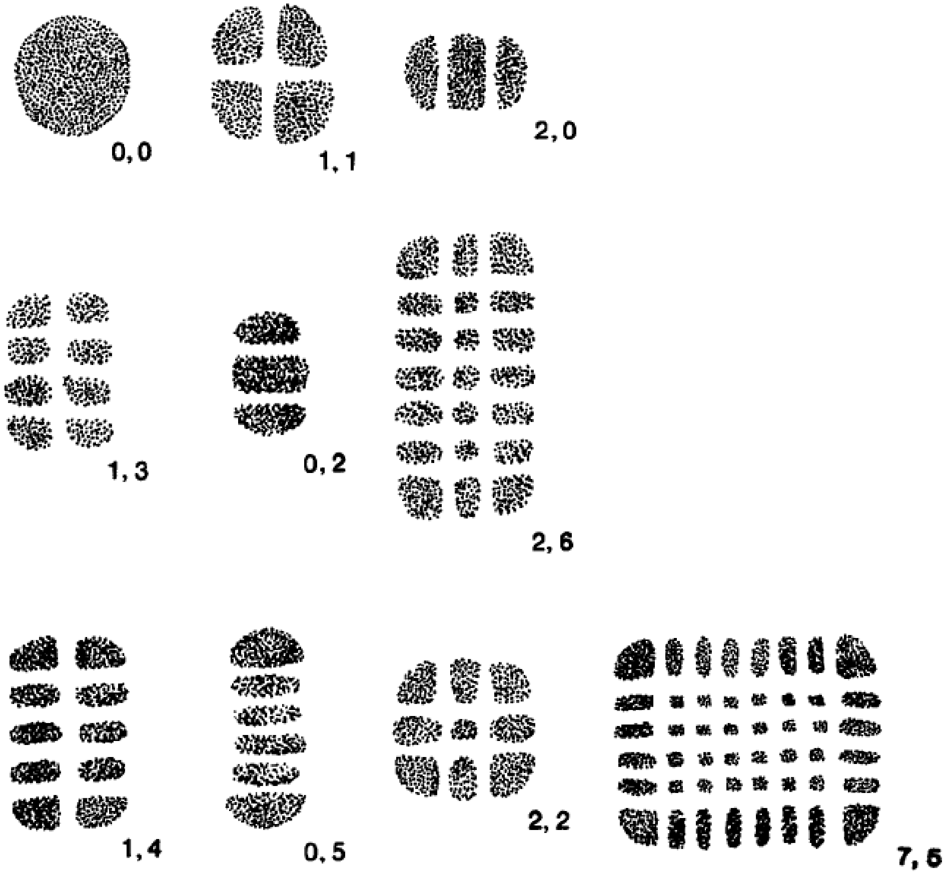




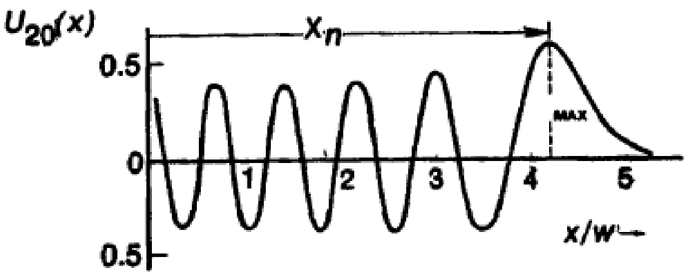
Higher-Order Gaussian Modes

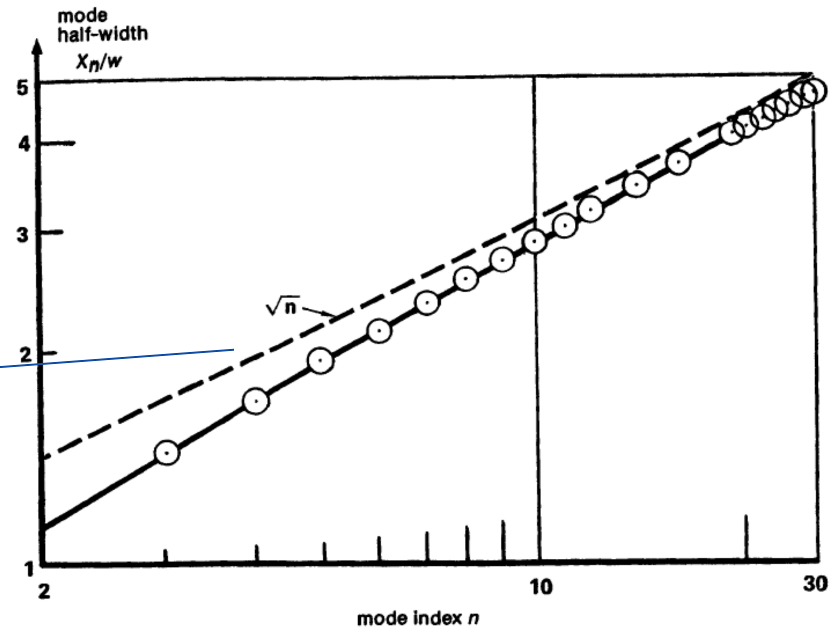
The intensity pattern of any given TEM_{nm} mode changes size but not shape as it propagates forward in z-a given TEM_{nm} mode looks exactly the same

Inherent property of the "Standard" Hermite-Gaussian solution



Higher-Order Mode Sizes





$$x_n \approx \sqrt{n} \times w$$

$$\Lambda_n \approx \frac{4w}{\sqrt{n}}$$ - spatial period of the ripples

- An aperture with radius a

$$x_n \leq a$$

$$n \leq N_{\max} \approx \left(\frac{a}{w}\right)^2$$ - works well for big *n* values

Common rule: $2a = \pi w$

Numerical Hermite-Gaussian Mode Expansion

$$f(x) = \sum_{n=0}^{N} c_n \tilde{u}_n(x; w), \qquad -a \le x \le a$$

$$c_n = \int_{-a}^{a} f(x)\, \tilde{u}_n^*(x)\, dx$$

→ **w, N - ?**

Numerical Hermite-Gaussian Mode Expansion

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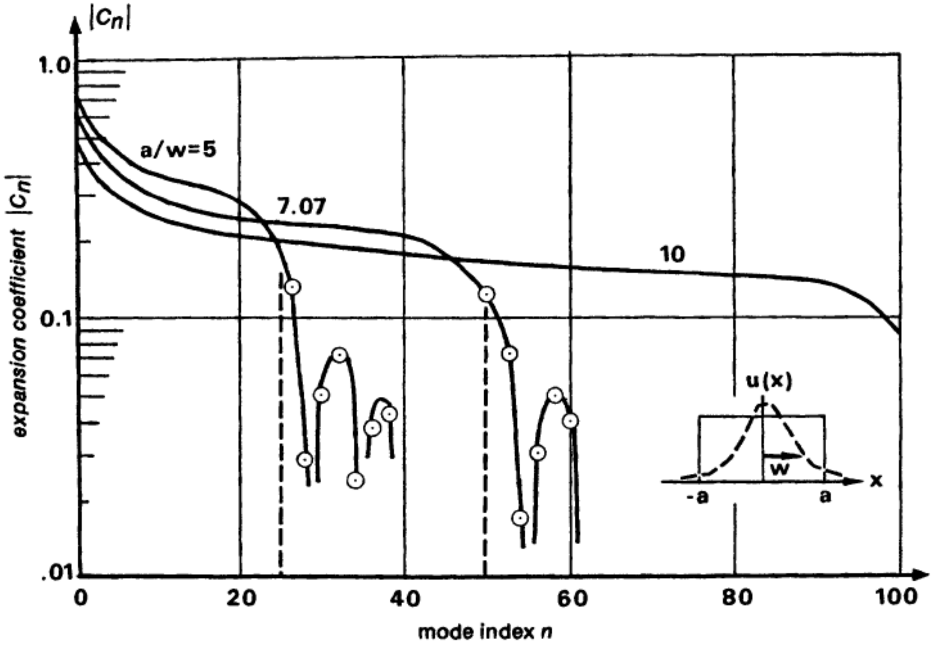
w, N - ?



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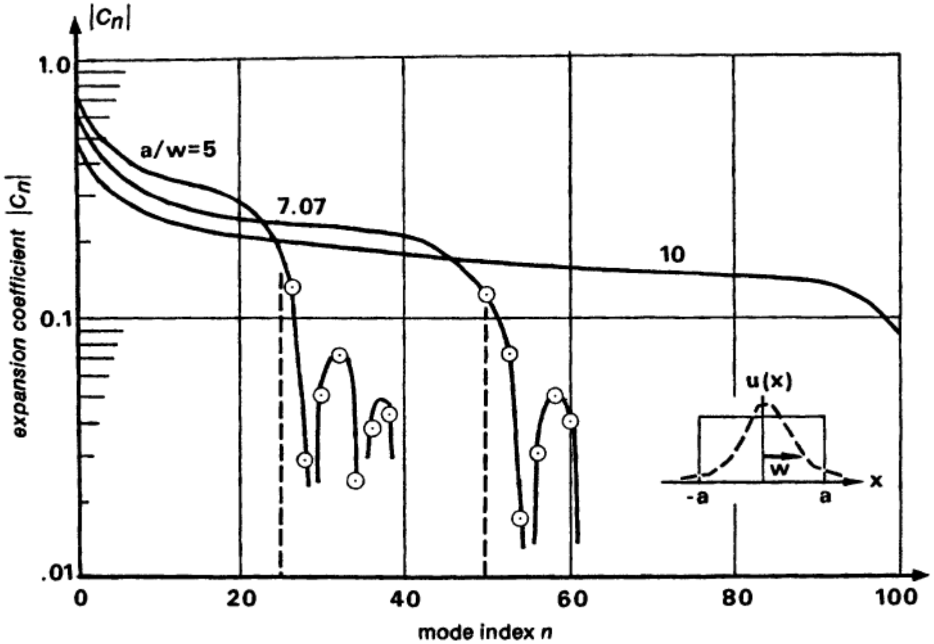
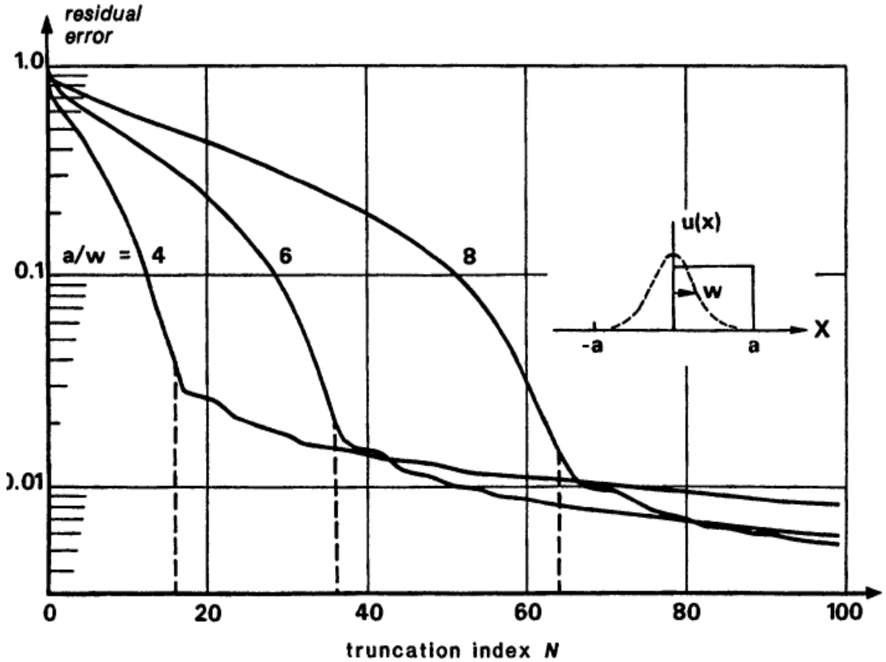
$$c_n = \int_{-a}^{a} f(x)\, \tilde{u}_n^*(x)\, dx$$

$$N \approx N_{\max} = (a/w)^2$$





Spatial Frequency Considerations

Expand arbitrary function f(x) across an aperture 2a with a finite sum of N+1 gaussian modes $\tilde{u}_n(x;w)$: w, N_{max} - ?

1. Calculate maximum spatial frequency of fluctuations in the function f(x)

   variations slower than $\approx \cos 2\pi x/\Lambda$

2. Select w, N so that the highest order TEM_N:
   - at least fill the aperture $N \geq N_{\max} \equiv \left(\frac{a}{w}\right)^2$
   - handle the highest spatial variation in the signal $\Lambda_N \approx \frac{4w}{\sqrt{N}} \leq \Lambda$

⇩

$$w \leq \sqrt{\frac{a\Lambda}{4}} \qquad N \geq \frac{4a}{\Lambda}$$

Thank you for the attention!