Transformation of Zernike coefficients: scaled, translated, and rotated wavefronts with circular and elliptical pupils

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Zernike polynomials and their associated coefficients are commonly used to quantify the wavefront aberrations of the eye. When the aberrations of different eyes, pupil sizes, or corrections are compared or averaged, it is important that the Zernike coefficients have been calculated for the correct size, position, orientation, and shape of the pupil. We present the first complete theory to transform Zernike coefficients analytically with regard to concentric scaling, translation of pupil center, and rotation. The transformations are described both for circular and elliptical pupils. The algorithm has been implemented in MATLAB, for which the code is given in an appendix. © 2007 Optical Society of America

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1. INTRODUCTION

The optical quality of human eyes is often described in terms of wavefront aberrations, represented by Zernike polynomials and their associated coefficients. The values of the Zernike coefficients will vary not only with the amount of aberrations but also with the size, position, orientation, and shape of the pupil. This paper presents a theory to analytically transform the Zernike coefficients for arbitrary scaling, centering, and rotation of circular and elliptical pupils.

The interest for measurements of optical wavefront aberrations of the human eye is growing both within research and industry. The result of a measurement is a sampled map of either the wavefront gradient or the wavefront height itself.[1] A convenient and popular method to present the measured wavefront is to reconstruct or expand the wavefront height with Zernike polynomials.[2–4] The Zernike polynomials constitute a complete, orthogonal set of functions defined over the unit circle. Each polynomial describes a mathematical wavefront shape, or aberration, and the associated Zernike coefficient gives the weight of that aberration in the total wavefront map.

The numerical values of the Zernike coefficients depend on the size of the measured pupil, and therefore Zernike coefficients have to be transformed between different pupils. A recalculation is, for example, necessary when Zernike coefficients for different pupil sizes are to be compared or averaged and when the optical aberrations for light-adapted pupils are to be found from wavefront measurements performed with dark-adapted or dilated pupils. Such a transformation to smaller pupil sizes has been the topic of four recent papers.[5–8] They all treat concentric contraction of circular pupils and reach the same result by slightly different methods. However, the pupil does not always contract concentrically; the center of the pupil can shift by as much as 0.4 mm.[9–11] Consequently, the change in centering should be taken into consideration when the Zernike coefficients are scaled for an accurate description of the smaller wavefront. The same problem is encountered in aberration-correcting contact lenses; the rotation and translation of the contact lens relative to the pupil will limit the benefits of the correction. Guirao *et al.* theoretically investigated the impact of translation and rotation on aberration correction by an approximated transformation of the Zernike coefficients.[12] Numerical methods, which translate the wavefront via a resampling process, have also been proposed.[2,13] However, a numerical approach does not explicitly give the relationship between the coefficients and the amount of translation.

In this paper we develop the analytical methodology of Campbell[6] further to include not only scaling but also translation and rotation of the wavefront. Additionally, we have included transformation of wavefronts with elliptical apertures, because the pupil will appear elliptical in shape, e.g., when measuring off axis. The Zernike coefficients for the elliptical pupil match the elliptical modification used by Atchison and Scott.[14] To our knowledge, the theory presented here is the first complete method for analytical transformations of Zernike coefficients with respect to scaling, translation, and rotation of circular and elliptical pupils. The following sections (2–7) explain the theory of the transformations and are not necessary for using the algorithm. If desired, the reader can therefore go directly to Appendix B and Section 8 for the MATLAB code and a short description of how it should be used, respectively.

2. COMPLEX MATRIX REPRESENTATION

The wavefront is expressed as a sum of Zernike coefficients, c_n^m, multiplied with their associated Zernike polynomial, Z_n^m. Here the indices n and m are the radial order,

$n=0\ldots n_{max}$, and the azimuthal frequency, $m=\pm n,\pm(n-2),\pm(n-4)\ldots$, respectively. This paper will use a modified and extended version of the matrix representation developed by Campbell.[6] The wavefront, $W(\rho,\theta)$, can be expressed as an inner product between $\langle Z|$, a row vector with the Zernike polynomials, and $|\mathbb{c}\rangle$, a column vector with the corresponding Zernike coefficients:

$$W(\rho,\theta)=\sum_n\sum_m \mathbb{c}_n^m Z_n^m(\rho,\theta)=\langle Z|\mathbb{c}\rangle,$$

$$\rho=\frac{physical\ radial\ coordinate}{maximum\ value\ of\ radial\ coordinate}=\frac{r}{r_0},\quad (1)$$

where ρ and θ are the radial and azimuthal coordinates, respectively, of the normalized entrance pupil of the eye[2,3] (θ is measured from the positive horizontal axis and is positive counter clockwise). The Zernike polynomials consist of a normalization factor, N_n, a radial polynomial, $R_n^m(\rho)$, and an angular function, $M^m(\theta)$:

$$Z_n^m(\rho,\theta)=N_nR_n^m(\rho)M^m(\theta),\quad (2)$$

$$N_n=\sqrt{n+1},\quad (3)$$

$$R_n^m(\rho)=\sum_{s=0}^{\frac{n-|m|}{2}}A_{n,s}^m\rho^{n-2s},$$

$$A_{n,s}^m=\frac{(-1)^s(n-s)!}{s!\,[0.5(n+|m|)-s]!\,[0.5(n-|m|)-s]!},\quad (4)$$

$$M^m(\theta)=e^{im\theta},\quad (5)$$

$$\int_0^1\int_0^{2\pi}Z_n^m(\rho,\theta)Z_j^{k*}(\rho,\theta)\rho\,\mathrm{d}\theta\,\mathrm{d}\rho =\pi\delta_{mk}\delta_{nj},\quad \delta_{ab}=\begin{cases}1 & a=b\\ 0 & a\neq b\end{cases}.\quad (6)$$

Equations (3) and (5) differ from the standard form;[2,3] here $M^m(\theta)$ is introduced as a complex function[4](instead of $\cos m\theta$ for $m\geq 0$ and $\sin m\theta$ for $m<0$), to simplify further calculations, and N_n is therefore also changed to fulfill the standard condition of orthogonality in Eq. (6) (*=complex conjugate). This means that the Zernike polynomials in this paper, $Z_n^m(\rho,\theta)$, are complex and that each complex polynomial $Z_n^m=Z_n^{-m*}$ corresponds to a linear combination of two real, standard Zernike polynomials of order $(n,+m)$ and $(n,-m)$. Also the coefficients, $\mathbb{c}_n^m$, of Eq. (1) are complex and differ from the standard. Because the wavefront is real, the complex coefficients must satisfy $\mathbb{c}_n^m=\mathbb{c}_n^{-m*}$, and each $\mathbb{c}_n^m$ is related to two real, standard Zernike coefficients, c_n^m, of order $(n,+m)$ and $(n,-m)$:

$$\mathbb{c}_n^m=(c_n^m-ic_n^{-m})/\sqrt{2},$$

$$\mathbb{c}_n^0=c_n^0,$$

$$\mathbb{c}_n^{-m}=(c_n^m+ic_n^{-m})/\sqrt{2},\quad m\ \text{positive}.\quad (7)$$

The conversion from complex Zernike coefficients back to the standard coefficients is given by

$$c_n^m=(\mathbb{c}_n^m+\mathbb{c}_n^{-m})/\sqrt{2},$$

$$c_n^0=\mathbb{c}_n^0,$$

$$c_n^{-m}=i(\mathbb{c}_n^m-\mathbb{c}_n^{-m})/\sqrt{2},\quad m\ \text{positive}.\quad (8)$$

A matrix representation will be introduced to simplify further calculations. However, to achieve block-diagonal matrices, one must first order the Zernike polynomials by increasing m; beginning with the most negative m-value, i.e., $m=-n_{\max}$ and, for each value of m, the terms are ordered from the lowest to the highest possible n value (see Appendix A for an explicit list of the following vectors and matrices for Zernike polynomials up to the third order):

$$\langle Z|=[Z_{n_{max}}^{-n_{max}}\quad Z_{n_{max}-1}^{-(n_{max}-1)}\quad Z_{n_{max}-2}^{-(n_{max}-2)}\quad Z_{n_{max}}^{-(n_{max}-2)}\quad \ldots\quad Z_{n_{max}}^{n_{max}}].\quad (9)$$

The row vector $\langle Z|$ with complex Zernike polynomials can now be rewritten as

$$\langle Z|=\langle\rho M|[R][N].\quad (10)$$

Here $\langle\rho M|$ is a new row vector containing terms with ρ raised to the power of the radial index n multiplied with the angular function $M^m(\theta)$, resulting in $\rho^n e^{im\theta}$ terms. The indices n and m for a term are the same as for the Zernike polynomial in the corresponding position of $\langle Z|$ in Eq. (9):

$$\langle\rho M|=[\rho^{n_{max}}e^{-in_{max}\theta}\quad \rho^{n_{max}-1}e^{-i(n_{max}-1)\theta}\quad \rho^{n_{max}-2}e^{-i(n_{max}-2)\theta}\quad \rho^{n_{max}}e^{-i(n_{max}-2)\theta}\ldots\rho^{n_{max}}e^{in_{max}\theta}].\quad (11)$$

The matrix $[R]$ is a square block-diagonal matrix, and $[N]$ is a square diagonal matrix. The size of both matrices equals the length of $\langle Z|$, and the diagonal blocks/elements are arranged in the same order as the elements of $\langle Z|$. The nonzero elements of $[R]$ equal the constants, $A_{n,s}^m$, of the radial polynomial in Eq. (4), and the diagonal elements of $[N]$ are the normalization factors given by Eq. (3):

$$[R] = \begin{bmatrix} A_{n_{max},0}^{-n_{max}} & 0 & 0 & 0 & \cdots & 0 \\ 0 & A_{n_{max}-1,0}^{-(n_{max}-1)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_{(n_{max}-2),0}^{-(n_{max}-2)} & A_{n_{max},1}^{-(n_{max}-2)} & \cdots & 0 \\ 0 & 0 & 0 & A_{n_{max},0}^{-(n_{max}-2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_{n_{max},0}^{n_{max}} \end{bmatrix}, \tag{12}$$

$$[N] = \begin{bmatrix} N_{n_{max}} & 0 & 0 & 0 & \cdots & 0 \\ 0 & N_{n_{max}-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & N_{(n_{max}-2)} & 0 & \cdots & 0 \\ 0 & 0 & 0 & N_{n_{max}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & N_{n_{max}} \end{bmatrix}. \tag{13}$$

The matrix representation introduced in this section is similar to the notation used by Campbell.[6] However, because Campbell only scaled the Zernike coefficients concentrically, the azimuthal coordinate, θ, was unaffected and the radial coordinate, ρ, could be transformed separately. This is not the case when arbitrary translation and rotation is performed, and we have therefore combined the radial and azimuthal coordinates in the complex vector $\langle \rho M|$.

3. TRANSFORMING ZERNIKE COEFFICIENTS

The matrix representation can be used to cut out an arbitrary part of the original wavefront to form a new wavefront, denoted by $'$. The matrices $[R]$ and $[N]$ contain constants and are unaltered, but the vectors $\langle \rho M|$ and $|c\rangle$ will be transformed. The wavefront itself, i.e., Eq. (10) into Eq. (1), has not changed, and therefore

$$\langle \rho M|[R][N]|c\rangle = \langle \rho M'|[R][N]|c'\rangle. \tag{14}$$

Suppose that the change in $\langle \rho M|$ is known as a transformation matrix $[\eta]$:

$$\langle \rho M| = \langle \rho M'|[\eta]. \tag{15}$$

Here $[\eta]$ will be a square matrix where each column describes how the corresponding term in $\langle \rho M|$ is transformed into $\langle \rho M'|$ terms. We would then be interested in a conversion matrix $[C]$, which gives the new set of Zernike coefficients from the original set:

$$|c'\rangle = [C]|c\rangle. \tag{16}$$

Inserting Eq. (15) and (16) into (14) gives

$$[\eta][R][N] = [R][N][C] \Rightarrow [C] = [N]^{-1}[R]^{-1}[\eta][R][N]. \tag{17}$$

Since $[N]$ and $[R]$ are known, $[C]$ and $|c'\rangle$ can be readily calculated if we know the transformation matrix $[\eta]$. The following sections will show how $[\eta]$ can be derived for scaling, translation, and rotation of wavefronts with circular and elliptical pupils. The size of $[\eta]$ will depend on the number of Zernike polynomials used and thus on the length of $\langle \rho M|$. Therefore, $[\eta]$ cannot be given explicitly. Instead, expressions will be derived for how an arbitrary element of $\langle \rho M|$ is transformed into $\langle \rho M'|$, which gives the columns of $[\eta]$ according to Eq. (15). It is useful to note that the elements of $\langle \rho M|$ can be rewritten as

$$\begin{aligned} \rho^n e^{im\theta} &= \rho^{n-m}(\rho e^{i\theta})^m = (\rho e^{i\theta}\rho e^{-i\theta})^{(n-m)/2}(\rho e^{i\theta})^m \\ &= (\rho e^{i\theta})^{(n+m)/2}(\rho e^{-i\theta})^{(n-m)/2}, \end{aligned} \tag{18}$$

where $(n+m)/2$ and $(n-m)/2$ are both integers (because $m = \pm n, \pm(n-2), \pm(n-4)\ldots$). The transformation of an element of $\langle \rho M|$ can therefore be derived if the transformation of $\rho e^{i\theta}$ is known. However, $\rho e^{i\theta} = \rho \cos\theta + i\rho \sin\theta$ can be considered as a complex coordinate, because its real and imaginary parts correspond to the horizontal and vertical axes, respectively, in the polar coordinate system of the pupil plane. Thus, the transformation of $\rho e^{i\theta}$ is a coordinate transformation. In the following sections we will therefore first derive the transformations in terms of $\rho e^{i\theta}$ and then in terms of $\langle \rho M|$ elements.

4. SCALING

Changing the size of a wavefront concentrically[5–8] from radius r_0 to r_s, as in the upper-left part of Fig. 1, is a simple scaling of the radial coordinate, ρ, with the scaling factor $\eta_s = r_s/r_0$, i.e., $\rho = \eta_s \rho'$, while the azimuthal coordinate is unchanged, $\theta = \theta'$:

$$\rho e^{i\theta} = \eta_s \rho' e^{i\theta'}. \tag{19}$$

Accordingly, the terms of $\langle \rho M|$ will change:

$$\rho^n e^{im\theta} = \eta_s^n \rho'^n e^{im\theta'}, \tag{20}$$

and $[\eta]$ will be a diagonal matrix with each element equal to η_s^n, where n is the exponent of the corresponding ρ term in $\langle \rho M|$ (see Appendix A).

5. TRANSLATION AND SCALING

Let the translation be described by the normalized polar coordinates η_t and θ_t, where $\eta_t = r_t/r_0$ according to the upper-right part of Fig. 1. When scaling is included (described by $\eta_s = r_s/r_0$, $\eta_s = 1$ means no scaling), the coordinate transformation is

$$\rho e^{i\theta} = \eta_s \rho' e^{i\theta'} + \eta_t e^{i\theta_t}. \tag{21}$$

This expression requires some manipulations before it can be written as a transformation of an arbitrary $\langle \rho M|$ element in the same manner as Eq. (20). First, insert Eq. (21) and its complex conjugate into Eq. (18). Because $(n+m)/2$ and $(n-m)/2$ are always integers, the binomial theorem

$$(a+b)^j = \sum_{k=0}^{j} \binom{j}{k} a^{j-k} b^k \tag{22}$$

can be used twice to give

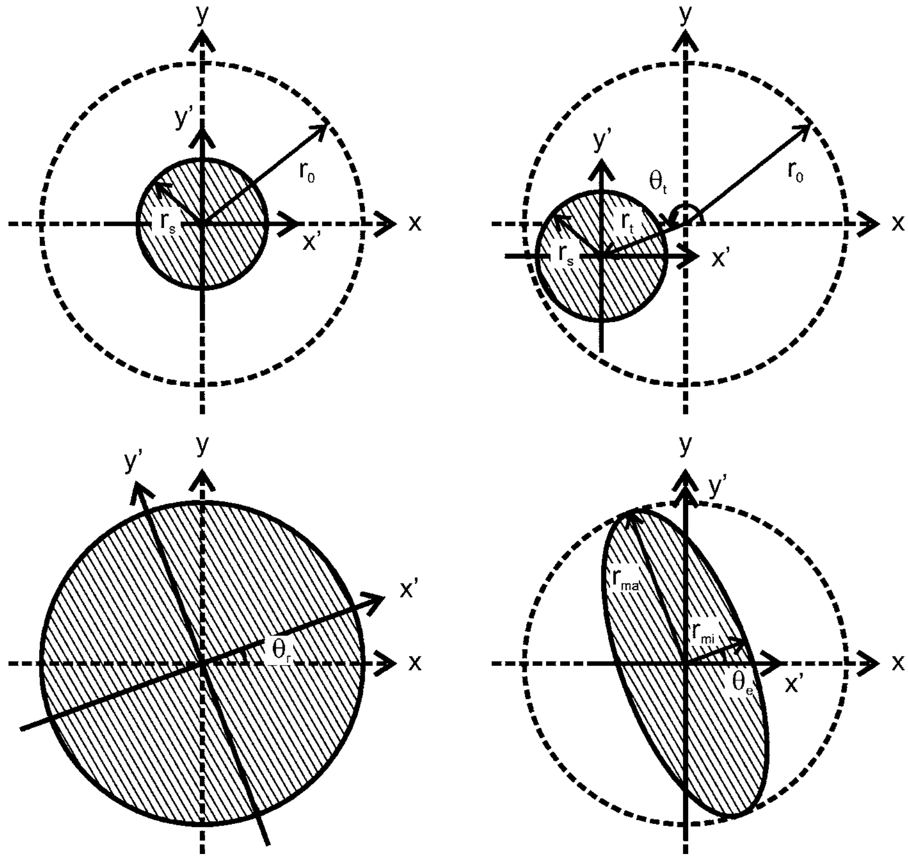

Fig. 1. The four possible coordinate transformations: (upper left) scaling from the original pupil size, r_0, to the new radius, r_s; (upper right) scaling combined with translation by r_t and θ_t; (lower left) rotation by the angle θ_r; (lower right) transformation to an ellipse, which is rotated by an angle θ_e, with the major radius, r_{ma}, equal to the original radius and a reduced minor radius, r_{mi}. The angles are positive counter clockwise. Dotted lines and circles show the original coordinate axes (x and y) and wavefronts, solid lines are the new axes (x' and y'), and striped areas are the new wavefronts.

$$\rho^n e^{im\theta} = \sum_{p=0}^{(n+m)/2} \sum_{q=0}^{(n-m)/2} \begin{pmatrix} \dfrac{n+m}{2} \\ p \end{pmatrix} \times \begin{pmatrix} \dfrac{n-m}{2} \\ q \end{pmatrix} \eta_s^{n-p-q} \eta_t^{p+q} e^{i(p-q)\theta_t} \rho'^{(n-p-q)} e^{i(m-p+q)\theta'}. \tag{23}$$

It is now possible to form the columns of the $[\eta]$ matrix one by one; Eq. (23) identifies the coefficients of $\langle \rho M'|$ terms with radial orders $n' = n-p-q$ and azimuthal frequencies $m' = (m-p+q)$ (see Appendix A).

6. ROTATION

A pure rotation of the wavefront will affect only the angular coordinate $\theta = \theta' + \theta_r$:

$$\rho e^{i\theta} = \rho' e^{i(\theta' + \theta_r)}, \tag{24}$$

where θ_r is the angle of rotation shown in the lower-left part of Fig. 1. Therefore

$$\rho^n e^{im\theta} = e^{im\theta_r} \rho'^n e^{im\theta'}, \tag{25}$$

which means that each element of $\langle \rho M|$ will couple to itself with an additional constant of $e^{im\theta_r}$. The matrix $[\eta]$ will thus be diagonal (see Appendix A).

7. ELLIPTICALLY SHAPED PUPILS

Zernike polynomials constitute an orthogonal base over the unit circle, and a wavefront described by Zernike coefficients should therefore preferably be sampled evenly within a circle. When the wavefront measurement is performed off axis, this is not possible since the pupil will appear elliptic and, thus, the Zernike polynomials and their associated coefficients will describe additional, extrapolated parts of the wavefront. Atchison and Scott therefore proposed that wavefronts for elliptical pupils should be stretched to convert the ellipse into a circle before fitting Zernike coefficients.[14] In this case the resulting Zernike coefficients will describe a version of the original wavefront stretched over a circular pupil. This stretching has the advantage that the square root of the sum of the squared Zernike coefficients will equal the true root-mean-square wavefront error, which is not the case when a circular pupil that encircles the ellipse is used.

The Zernike coefficients for a stretched elliptical pupil can be found from the coefficients of an encircling circular pupil; i.e., the elliptical pupil is part of the circular pupil and the major radius of the ellipse equals the radius of the circle. The minor radius of the ellipse equals a fraction η_e of the circle radius:

$$\eta_e = \frac{minor\ radius}{major\ radius} = \frac{r_{mi}}{r_{ma}},$$

as shown in the lower-right part of Fig. 1. The one-dimensional stretching of the coordinate system has the same effect as a scaling of the wavefront in one direction. Let the angle θ_e, measured from the horizontal coordinate axis to the minor axis of the ellipse, denote the direction of the stretching. This leads to the coordinate transformation

$$\begin{aligned}\rho e^{i\theta} &= e^{i\theta_e}(\rho\cos(\theta-\theta_e) + i\rho\sin(\theta-\theta_e)) \\ &= e^{i\theta_e}(\eta_e\rho'\cos(\theta'-\theta_e) + i\rho'\sin(\theta'-\theta_e)) \\ &= \frac{\eta_e+1}{2}\rho' e^{i\theta'} + \frac{\eta_e-1}{2}e^{i2\theta_e}\rho' e^{-i\theta'}. \end{aligned} \tag{26}$$

Following the same steps as for translation, Eq. (26) and its complex conjugate is inserted into Eq. (18) and the binomial theorem (22) is used twice:

$$\begin{aligned}\rho^n e^{im\theta} = \frac{1}{2^n}\sum_{p=0}^{(n+m)/2}\sum_{q=0}^{(n-m)/2}\begin{pmatrix}\dfrac{n+m}{2}\\ p\end{pmatrix}\begin{pmatrix}\dfrac{n-m}{2}\\ q\end{pmatrix}(\eta_e+1)^{n-p-q} \\ \times(\eta_e-1)^{p+q}e^{i2(p-q)\theta_e}\rho'^n e^{i(m-2p+2q)\theta'}. \end{aligned} \tag{27}$$

This expression can now be used to derive the columns of the $[\eta]$ matrix in the same way as for the previous sections; the coefficients of $\langle\rho M'|$ terms with radial orders $n'=n$ and azimuthal frequencies $m'=(m-2p+2q)$ can be identified.

The elliptically stretched Zernike coefficients derived with Eq. (27) can be transformed in a manner similar to Zernike coefficients for unstretched wavefronts described in Sections 4–6. Equation (20) can be used directly to scale coefficients for stretched wavefronts. However, translation of elliptically stretched coefficients according to Eq. (23) will have different values of r_t depending on the direction of the translation, θ_t. Therefore, the translation should preferably be performed before the elliptical scaling. The rotation of Eq. (25) also applies to Zernike coefficients of elliptically stretched wavefronts; both the wavefront and the elliptical shape of the pupil will rotate.

8. SUMMARY

In this paper a theory for wavefront manipulations directly on the Zernike coefficients has been described. Wavefronts with circular and elliptical pupils can be rapidly scaled, translated, and rotated arbitrary amounts without having to resample the wavefront. Note that the transformed wavefront should not be extended outside the border of the original wavefront data to avoid extrapolation errors.

The algorithm has been implemented in MATLAB, and the code is given in Appendix B (elliptical scaling is not included; it can, however, be obtained from the authors). In the code, the output is the vector, $C2$, containing the new pupil diameter in millimeters as the first term followed by the transformed standard[2] Zernike coefficients in micrometers. The input parameters are $C1$, the vector with the original pupil diameter in millimeters ($dia1$) followed by the standard[2] Zernike coefficients of the original wavefront given in micrometers; $dia2$, the desired diameter in millimeters for concentric scaling ($\eta_s=dia2/dia1$); tx and ty, the translation in Cartesian coordinates in millimeters ($\eta_t=2(tx^2+ty^2)^{1/2}/dia1, \theta_t=\text{atan}(ty/tx)$); and

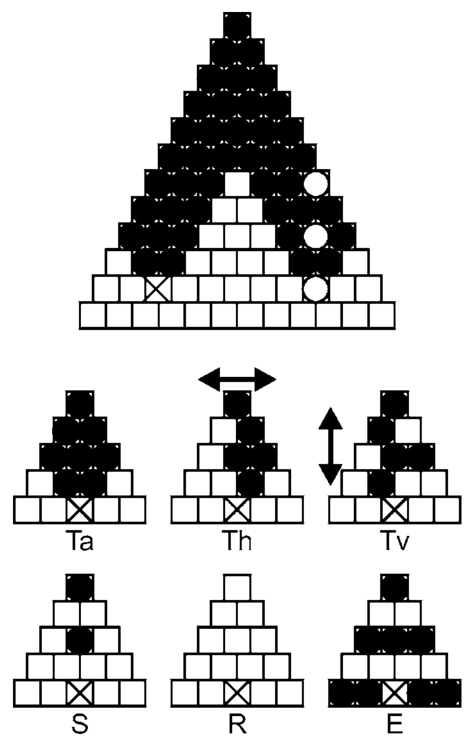

Fig. 2. Coupling of Zernike coefficients when the wavefront is transformed. The square boxes of the pyramids represent real, standard Zernike coefficients with the radial order, n, increasing downward and the azimuthal frequency, m, going from negative to positive values from left to right. The crosses denote the single original Zernike coefficient before the transformation, and the filled circles denote the coefficients to which the original coefficient couple. The large, upper pyramid shows as an example how the coefficient with $n=10$ and $m=-6$ is transformed when the wavefront is arbitrarily translated and scaled; the unfilled circles denote the additional coefficients if rotation also is included. The six small pyramids show the transformation of spherical aberration: Ta, arbitrary translation; Th, horizontal translation; Tv, vertical translation; S, concentric scaling; R, rotation; E, elliptical scaling.

thetaR, the angle of rotation measured in degrees counter clockwise from the horizontal coordinate axis as in Fig. 1 (corresponding to θ_r). The code will first scale and translate the wavefront, and the rotation is then performed around the new pupil center. If an alternative order of the transformations is desired, the code has to be used multiple times, one for each transformation. Note that the algorithm corresponds to a *coordinate transformation*; i.e., the wavefront is fixed, whereas the coordinate axes are moved and rotated as shown in Fig. 1.

The MATLAB code in Appendix B starts by creating the matrices $[R]$ and $[N]$. Also used is a permutation matrix $[P]$, which converts the sorting of the Zernike polynomials from the standard order[2,3] to the order used in this paper and by Campbell.[6] The matrix $[\eta]$ is formed successively column by column with the function *transform*, which performs the calculations of Eqs. (20), (23), and (25). The implemented code uses complex Zernike coefficients for the calculations and includes Eqs. (7) and (8) to convert between complex and real coefficients. This means that the input and output Zernike coefficients of the algorithm are the conventional Zernike coefficients ordered according to the standard.[2]

With this formalism we can derive some characteristics of how the individual Zernike coefficients couple to each other when the wavefront is manipulated. Specifically, we can investigate how one single, nonzero coefficient is transformed, which is shown schematically in Fig. 2. For concentric reduction of the size of the wavefront, the original Zernike coefficient will decrease and couple to Zernike coefficients of the same azimuthal frequency, m, but of lower radial order, n. If the wavefront is rotated, rotationally symmetric aberrations (i.e., with $m=0$) will remain unchanged and a coefficient with $m\neq 0$ will couple between itself (n,m) and the mirror coefficient $(n,-m)$. Translation, without reduction in size, will not change the value of the original coefficient. The coefficient will couple only to lower radial orders, n, and the coupling will depend on the direction of the translation. Elliptical scaling reduces the original coefficient and gives coupling to other coefficients of the same radial order, n, and to coefficients with lower radial orders. In summary, this means that a Zernike polynomial never affects a polynomial of radial order higher than the original n.

APPENDIX A: WORKED-OUT EXAMPLE UP TO THIRD ORDER $(n=0\ldots 3, m=\pm n, \pm(n-2),\ldots)$

$$\langle Z| = [Z_3^{-3}\quad Z_2^{-2}\quad Z_1^{-1}\quad Z_3^{-1}\quad Z_0^{0}\quad Z_2^{0}\quad Z_1^{1}\quad Z_3^{1}\quad Z_2^{2}\quad Z_3^{3}] = [2\rho^3 e^{-i3\theta}\quad \sqrt{3}\rho^2 e^{-i2\theta}\quad \sqrt{2}\rho e^{-i\theta}\quad 2(3\rho^3-2\rho)e^{-i\theta}\quad 1\quad \sqrt{3}(2\rho^2-1)\quad \sqrt{2}\rho e^{i\theta}\quad 2(3\rho^3-2\rho)e^{i\theta}\quad \sqrt{3}\rho^2 e^{i2\theta}\quad 2\rho^3 e^{i3\theta}], \tag{A1}$$

$$\langle \rho M| = [\rho^3 e^{-i3\theta}\quad \rho^2 e^{-i2\theta}\quad \rho e^{-i\theta}\quad \rho^3 e^{-i\theta}\quad 1\quad \rho^2\quad \rho e^{i\theta}\quad \rho^3 e^{i\theta}\quad \rho^2 e^{i2\theta}\quad \rho^3 e^{i3\theta}], \tag{A2}$$

$$[R] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{A3}$$

$$[N] = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \tag{A4}$$

$$|\mathbb{c}\rangle = \begin{bmatrix} \mathbb{c}_3^{-3} \\ \mathbb{c}_2^{-2} \\ \mathbb{c}_1^{-1} \\ \mathbb{c}_3^{-1} \\ \mathbb{c}_0^{0} \\ \mathbb{c}_2^{0} \\ \mathbb{c}_1^{1} \\ \mathbb{c}_3^{1} \\ \mathbb{c}_2^{2} \\ \mathbb{c}_3^{3} \end{bmatrix} = \begin{bmatrix} (c_3^3 + ic_3^{-3})/\sqrt{2} \\ (c_2^2 + ic_2^{-2})/\sqrt{2} \\ (c_1^1 + ic_1^{-1})/\sqrt{2} \\ (c_3^1 + ic_3^{-1})/\sqrt{2} \\ c_0^0 \\ c_2^0 \\ (c_1^1 - ic_1^{-1})/\sqrt{2} \\ (c_3^1 - ic_3^{-1})/\sqrt{2} \\ (c_2^2 - ic_2^{-2})/\sqrt{2} \\ (c_3^3 - ic_3^{-3})/\sqrt{2} \end{bmatrix}, \tag{A5}$$

$$[\eta]_{scale} = \begin{bmatrix} \eta_s^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta_s^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \eta_s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_s^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_s^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \eta_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_s^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_s^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_s^3 \end{bmatrix}, \tag{A6}$$

$$[\eta]_{translate} = \begin{bmatrix} \eta_s^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3\eta_s^2\eta_t e^{-i\theta_t} & \eta_s^2 & 0 & \eta_s^2\eta_t e^{i\theta_t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 3\eta_s\eta_t^2 e^{-i2\theta_t} & 2\eta_s\eta_t e^{-i\theta_t} & \eta_s & 2\eta_s\eta_t^2 & 0 & \eta_s\eta_t e^{i\theta_t} & 0 & \eta_s\eta_t^2 e^{i2\theta_t} & 0 & 0 \\ 0 & 0 & 0 & \eta_s^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \eta_t^3 e^{-3i\theta_t} & \eta_t^2 e^{-2i\theta_t} & \eta_t e^{-i\theta_t} & \eta_t^3 e^{-i\theta_t} & 1 & \eta_t^2 & \eta_t e^{i\theta_t} & \eta_t^3 e^{i\theta_t} & \eta_t^2 e^{i2\theta_t} & \eta_t^3 e^{i3\theta_t} \\ 0 & 0 & 0 & 2\eta_s^2\eta_t e^{-i\theta_t} & 0 & \eta_s^2 & 0 & 2\eta_s^2\eta_t e^{i\theta_t} & 0 & 0 \\ 0 & 0 & 0 & \eta_s\eta_t^2 e^{-i2\theta_t} & 0 & \eta_s\eta_t e^{-i\theta_t} & \eta_s & 2\eta_s\eta_t^2 & 2\eta_s\eta_t e^{i\theta_t} & 3\eta_s\eta_t^2 e^{i2\theta_t} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_s^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_s^2\eta_t e^{-i\theta_t} & \eta_s^2 & 3\eta_s^2\eta_t e^{i\theta_t} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \eta_s^3 \end{bmatrix}, \tag{A7}$$

$$[\eta]_{rotate} = \begin{bmatrix} e^{-i3\theta_r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-i2\theta_r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\theta_r} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\theta_r} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i\theta_r} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\theta_r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i2\theta_r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i3\theta_r} \end{bmatrix}. \tag{A8}$$

APPENDIX B: MATLAB CODE

```
function C2=TransformC(Cl,dia2,tx,ty,thetaR)
% "TransformC" returns transformed Zernike coefficient set, C2, from the original set, C1,
% both in standard ANSI order, with the pupil diameter in mm as the first term.
% dia2—new pupil diameter [mm]
% tx, ty—Cartesian translation coordinates [mm]
% thetaR—angle of rotation [degrees]
% Scaling and translation is performed first and then rotation.
dial=C1(1); % Original pupil diameter
C1=C1(2:end);
```

```
etaS=dia2/dia1;  % Scaling factor
etaT=2*sqrt(tx^2+ty^2)/dial;  % Translation coordinates
thetaT=atan2(ty,tx);
thetaR=thetaR*pi/180;   % Rotation in radians
jnm=length(C1)-1; nmax=ceil((-3+sqrt(9+8*jnm))/2); jmax=nmax*(nmax+3)/2;
S=zeros(jmax+1,1); S(1:length(C1))=C1; C1=S; clear S
P=zeros(jmax+1);  % Matrix P transforms from standard to Campbell order
N=zeros(jmax+1);  % Matrix N contains the normalization coefficients
R=zeros(jmax+1);  % Matrix R is the coefficients of the radial polynomials
CC1=zeros(jmax+1,1);  % CC1 is a complex representation of C1
counter=1;
for m=-nmax:nmax   % Meridional indexes
  for n=abs(m):2:nmax  % Radial indexes
    jnm=(m+n*(n+2))/2;
    P(counter,jnm+1)=1;
    N(counter,counter)=sqrt(n+1);
    for s=0:(n-abs(m))/2
      R(counter-s,counter)=(-1)^s*factorial(n-s)/(factorial(s)*factorial((n+m)/2-s)*
  factorial((n-m)/2-s));
    end
    if m<0, CC1(jnm+1)=(C1((-m+n*(n+2))/2+1)+i*C1(jnm+1))/sqrt(2);
    elseif m==0, CC1(jnm+1)=C1(jnm+1);
    else, CC1(jnm+1)=(C1(jnm+1)-i*C1((-m+n*(n+2))/2+1))/sqrt(2);end
    counter=counter+1;
end, end
ETA=[];  % Coordinate-transfer matrix
for m=-nmax:nmax
  for n=abs(m):2:nmax
    ETA=[ETA P*(transform(n,m,jmax,etaS,etaT,thetaT,thetaR))];
end, end
C=inv(P)*inv(N)*inv(R)*ETA*R*N*P;
CC2=C*CC1;
C2=zeros(jmax+1,1);  % C2 is formed from the complex Zernike coefficients, CC2
for m=-nmax:nmax
  for n=abs(m):2:nmax
    jnm=(m+n*(n+2))/2;
    if m<0, C2(jnm+1)=imag(CC2(jnm+1)-CC2((-m+n*(n+2))/2+1))/sqrt(2);
    elseif m==0, C2(jnm+1)=real(CC2(jnm+1));
    else, C2(jnm+1)=real(CC2(jnm+1)+CC2((-m+n*(n+2))/2+1))/sqrt(2);
end, end, end
C2=[dia2;C2];
%
function Eta=transform(n,m,jmax,etaS,etaT,thetaT,thetaR)
% Returns coefficients for transforming a ro^n*exp(i*m*theta)-term into '-terms
Eta=zeros(jmax+1,1);
for p=0:((n+m)/2)
  for q=0:((n-m)/2)
    nnew=n-p-q;  mnew=m-p+q;
    jnm=(mnew+nnew*(nnew+2))/2;
    Eta(floor(jnm+1))=Eta(floor(jnm+1))+nchoosek((n+m)/2,p)*nchoosek((n-m)/2,q)
* etaS^(n-p-q)*etaT^(p+q)*exp(i*((p-q)*(thetaT-thetaR)+m*thetaR));
end, end
```

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